On sufficient conditions for starlikeness of p-valently Bazilevič functions of the type $\beta$ and order $\gamma$ \footnote{Received 15 September, 2008 \newline Accepted for publication (in revised form) 10 October, 2008}

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Abstract

The aim of this paper is to establish certain sufficient conditions for the class of starlikeness p-valently Bazilevič functions of the type $\beta$ and order $\gamma$. Our result is of general nature and capable of yielding a number of unknown new and interesting results.

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1 Introduction and definitions

Let \( \mathcal{A}_n(p) \) be the class of normalized functions of the form

\[
(1) \quad f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\})
\]

which are analytic in the unit disk \( \Delta := \{z : |z| < 1\} \). A function \( f \in \mathcal{A}_n(p) \) is said to be in the class \( \mathcal{S}^*_n(p, \alpha) \), if it satisfies

\[
(2) \quad \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; \ z \in \Delta).
\]

A function in the class \( \mathcal{S}^*_n(\alpha) \) is starlike of order \( \alpha \) in \( \Delta \).

Let \( \mathcal{K}_n(\alpha) \) be subclass of \( \mathcal{A}_n(p) \) consisting of functions \( f(z) \) which satisfies

\[
(3) \quad \text{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \alpha \quad (0 \leq \alpha < p; \ z \in \Delta),
\]

where \( g(z) \in \mathcal{S}_n(p, 0) := \mathcal{S}_n(p) \).

Also a function \( f \in \mathcal{A}_n \) is said to be p-valently Bazilević function of type \( \beta \) \( (\beta \geq 0) \) and order \( \gamma \) \( (0 \leq \gamma < p) \), if there exists a function \( g(z) \in \mathcal{S}^*_n(p) \), such that

\[
(4) \quad \text{Re}\left\{ \frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} \right\} > \gamma \quad (0 \leq \gamma < p; \ z \in \Delta).
\]

We denote the class of all such functions by \( \mathcal{B}_n(\beta, \gamma) \). In particular, when \( \beta = 1 \), a function \( f \in \mathcal{K}_n(\gamma) := \mathcal{B}_n(1, \gamma) \) is p-valently close-to-convex of order \( \gamma \) in \( \Delta \). Moreover if \( \beta = 0 \), then \( \mathcal{B}_n(0, \gamma) := \mathcal{S}^*_n(\gamma) \).

In order to prove our main results, we shall require the following lemma.
Lemma 1 ([3]) Let \( \Omega \) be a set in the complex plane \( \mathbb{C} \) and suppose that \( \phi \) is a mapping from \( \mathbb{C}^2 \times \triangle \) to \( \mathbb{C} \) which satisfies \( \phi(ix; y; z) \notin \Omega \) for \( z \in \triangle \) and for all real \( x \) and \( y \) such that \( y \geq -n(1+x^2)/2 \). If the function \( P(z) = 1 + c_n z^n + \ldots \), is analytic in \( \triangle \) and \( \phi(P(z); zp(z); z) \in \Omega \) all \( z \in \triangle \) then \( \Re P(z) > 0 \).

2 Main Results

Applying Lemma 1, we now derive the following

Theorem 1 Let \( f(z) \in A_n(p) \), satisfies

\[
\Re \left\{ \frac{\alpha z f'(z)}{[f(z)]^{1-\mu} [g(z)]^\mu} + \frac{\alpha z f''(z)}{f'(z)} - \alpha (1-\mu) \frac{zf''(z)}{f'(z)} - \alpha \mu \frac{zg'(z)}{g(z)} + 1 \right\} \frac{zf'(z)}{[f(z)]^{1-\mu} [g(z)]^\mu} > \alpha \beta \left( \beta \delta + \frac{n}{2} - 1 \right) + \left( \beta - \frac{np\alpha}{2} \right) \quad (z \in \triangle, 0 \leq \alpha, \beta < p),
\]

then \( f(z) \in B_n(p, \mu, \beta) \).

Proof. Define \( P(z) \) by

\[
\frac{zf'(z)}{[f(z)]^{1-\mu} [g(z)]^\mu} = (p - \beta) P(z) + \beta,
\]
then \( P(z) = 1 + c_n z^n + \ldots \) is analytic in \( \triangle \). Differentiate logarithmically (6) with respect to \( z \) and making a little simplification, we get

\[
\frac{\delta zf'(z)}{[f(z)]^{1-\mu} [g(z)]^\mu} + \frac{zf''(z)}{f'(z)} - (1-\mu) \frac{zf'(z)}{f(z)} - \mu \frac{zg'(z)}{g(z)}
\]

\[
= \frac{(p - \beta) zP'(z) + \delta [(p - \beta) P(z) + \beta]^2}{(p - \beta) P(z) + \beta}
\]
\[
\left[ \frac{\alpha \delta z f'(z)}{f(z)^{1+\mu} g(z)^\mu} + \alpha \frac{zf''(z)}{f'(z)} - \alpha (1-\mu) \frac{zf'(z)}{f(z)} - \alpha \mu \frac{zg'(z)}{g(z)} + 1 \right] \frac{zf'(z)}{f(z)^{1+\mu} g(z)^\mu} \\
= \alpha (p-\beta) z P'(z) + (2\beta\alpha\delta - \alpha + 1) (p-\beta) P(z) + \alpha \delta (p-\beta)^2 P^2(z) + (\delta \alpha \beta^2 - \alpha \beta + \beta)
\]

(8) 

where

\[
\phi(r, s; t) = \alpha (p-\beta) \delta + (2\beta \alpha \delta - \alpha + 1) (p-\beta) r + \alpha \delta (p-\beta)^2 r^2 + (\delta \alpha \beta^2 - \alpha \beta + \beta)
\]

for all real \(x\) and \(y\) satisfying \(y = -n(1 + x^2)/2\), we have

\[
\text{Re} \phi(ix, y; z) = \alpha (p-\beta) y - \alpha \delta (p-\beta)^2 x^2 + \beta (\alpha \beta \delta - \alpha + 1)
\]

\[
\leq -\frac{\alpha}{2} (p-\beta) n - \left\{ \alpha \delta (p-\beta)^2 + \frac{n\alpha}{2} (p-\beta) \right\} x^2 + \beta (\alpha \beta \delta - \alpha + 1)
\]

\[
= -\frac{\alpha}{2} (p-\beta) n - \frac{\alpha (p-\beta)}{2} \left( n + 2p\delta - 2\beta \right) x^2 + \beta (\alpha \beta \delta - \alpha + 1)
\]

(10) 

\[
\leq \alpha \beta \left( \beta \delta + \frac{n}{2} - 1 \right) + \left( \beta - \frac{n\alpha}{2} \right).
\]

Let

\[
\Omega = \left\{ w; \text{Re} w > 0, \alpha \beta \left( \beta \delta + \frac{n}{2} - 1 \right) + \left( \beta - \frac{n\alpha}{2} \right) \right\},
\]

then

\[
\phi(P(z), z P'(z); z) \in \Omega \quad \text{and} \quad \phi(ix, y; z) \notin \Omega.
\]

For all real \(x\) and \(y = -n(1 + x^2)/2, z \in \Delta\), By application of lemma (1), the result (5) follows at once.

On taking \(\delta = 1, \mu = 0\) in Theorem 1, we get
Corollary 1 If \( f(z) \in A_n(p) \) satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{\alpha z f''(z)}{f'(z)} + 1 \right) \right\} > \alpha \beta \left( \beta + \frac{n}{2} - 1 \right) + \left( \beta + \frac{n p \alpha}{2} \right),
\]

\((z \in \Delta, 0 \leq \alpha, 0 \leq \beta < p)\) then \( f(z) \in S_n(p, \beta) \).

On taking \( \beta = 0, n = 1, p = 1 \) and \( \beta = \alpha/2, n = 1, p = 1 \) respectively we have a known result obtained by Ravichandran et al. [1]. On taking \( \mu = 1 \) and \( \delta = 1 \), we find an interesting result contained in the following corollary.

Corollary 2 If \( f(z) \in A_n(p) \), satisfies

\[
\text{Re} \left\{ \frac{\alpha z f'(z)}{g(z)} + \frac{\alpha z f''(z)}{f'(z)} - \alpha \frac{z g'(z)}{g(z)} + 1 \right\} \frac{z f'(z)}{g(z)} > \alpha \beta \left( \beta + \frac{n}{2} - 1 \right) + \left( \beta - \frac{n p \alpha}{2} \right)
\]

\((z \in \Delta, 0 \leq \alpha, \beta < p)\),

then \( f(z) \in K_n(p, \beta) \).

Theorem 2 Let \( 0 \leq \beta < p \),

\[
\lambda = \left( p - \beta + \frac{n}{2} \right)^2 (p - \beta)^2, \quad \eta = \left\{ \frac{n}{2} (p - \beta) - (\delta \beta^2 - \beta) \right\}^2
\]

\( v = (p - \beta)^2 + (\delta \beta^2 - \beta) \) and \( \sigma = (p - \beta)^2 (2 \beta - 1)^2 \).

Also, suppose that \( t_0 \) to be the smallest positive root of the equation

\[
2 \lambda (p - \beta)^2 t^3 + \left\{ (p - \beta)^2 (2 \lambda + \eta - v + \sigma) + 3 \lambda \beta^2 \right\} t^2 + 2 \beta^2 (2 \lambda + \eta - v + \sigma) t + (\lambda + 2 \eta - v + \sigma) \beta^2 - (p - \beta)^2 \eta = 0
\]

and

\[
\frac{(p - \beta)^2 (1 + t_0)}{(p - \beta)^2 t_0 + \beta^2} \left[ \lambda t_0^2 + (\lambda + \eta - v + \sigma) t_0 + \eta \right] = \rho^2.
\]
Now, if
\[ \left| \frac{\delta z f'(z)}{f(z) \alpha} - p \right| \leq |z| \left| \frac{\delta f'(z)}{f(z) \alpha} \right| + \left| \frac{zf''(z)}{f'(z) \alpha} - \left( 1 - \mu \right) \frac{zf'(z)}{f(z) \alpha} - \mu \frac{zg'(z)}{g(z) \alpha} \right| \leq \mu z \in \Delta \]
then \( f(z) \in \mathcal{B}_n(p, \mu, \beta) \).

**Proof.** Define \( P(z) \) by
\[ (p - \beta) P(z) + \beta = \left| \frac{zf'(z)}{f(z) \alpha} \right| \]
then \( P(z) = 1 + c_n z^n + \ldots \), is analytic in \( \Delta \). A computation shows that
\[ \frac{\delta z f'(z)}{f(z) \alpha} + \frac{zf''(z)}{f'(z) \alpha} - (1 - \mu) \frac{zf'(z)}{f(z) \alpha} - \mu \frac{zg'(z)}{g(z) \alpha} = \frac{\varphi}{(p - \beta) P(z) + \beta} \]
where
\[ \varphi(r, s; t) = \frac{(p - \beta)(r - 1)}{(p - \beta) r + \beta} [(p - \beta) s + \delta \{(p - \beta) r + \beta\}^2 - \{(p - \beta) r + \beta\}] \]
For all real \( x \) and \( y = -n(1 + x^2)/2 \), we have
\[ |\varphi(ix, y; z)| = \frac{(p - \beta)^2 (1 + x^2)}{(p - \beta)^2 x^2 + \beta^2} \left[ \{(p - \beta) y - (p - \beta)^2 x^2 \delta + \beta^2 - \beta\}^2 \right. \\
\left. + (p - \beta)^2 x^2 (2 \delta \beta - 1)^2 \right] = g(x^2, y) \]
Now
\[ \frac{\partial g}{\partial y} = \frac{2(p - \beta)^2 (1 + x^2)}{(p - \beta)^2 x^2 + \beta^2} [(p - \beta) y - (p - \beta)^2 x^2 \delta + \beta^2 - \beta] < 0 \]
then we have
\[ h(t) = g(t, -n(1 + t)/2) \leq g(t, y) \]
where

\[ h(t) = \frac{(p-\beta)^2}{(p-\beta)^2 + \beta^2} \left[ (p-\beta)^2 (p-\beta+\eta)^2 t^2 + \left\{ (p-\beta)^2 (p-\beta+\eta)^2 + \left( \frac{\eta}{2} (p-\beta) - \delta^2 - \beta \right)^2 \right\} t \right. \\
- \left. \left( (p-\beta)^2 + (\delta^2 - \beta) \right)^2 + (p-\beta)^2 (2\beta - 1)^2 \right\} t + \left( \frac{\eta}{2} (p-\beta) - \delta^2 - \beta \right)^2 \right] \\
= \frac{(p-\beta)^2(1+t)}{(p-\beta)^2t+\beta^2} \left[ \lambda t^2 + (\lambda + \eta - v + \sigma) t + \eta \right], \]

where

\[ \lambda = \left( p - \beta + \frac{\eta}{2} \right)^2 (p - \beta)^2, \quad \eta = \left\{ \frac{\eta}{2} (p - \beta) - \delta^2 - \beta \right\}^2 \]

and

\[ \nu = \left\{ (p - \beta)^2 + (\delta^2 - \beta) \right\}^2 \text{ and } \sigma = (p - \beta)^2 (2\beta - 1)^2 \]

Now

\[ h'(t) = \frac{(p-\beta)^2}{(p-\beta)^2 t + \beta^2} \left[ 2(p-\beta)^2 \lambda t^3 + \left\{ (p-\beta)^2 (2\lambda + \eta - v + \sigma) + 3\beta^2 \lambda \right\} t^2 \\
+ 2\beta^2 (2\lambda + \eta - v + \sigma) t + (\lambda + 2\eta - v + \sigma) \beta^2 - (p-\beta)^2 \eta \right] \]

Taking \( h(t) = 0 \), we obtain (13) which is cubic in \( t \). Let \( t_0 \) be the smallest positive root of the equation, then we have \( h(t) = h(t_0) \), and hence

\[ |\phi(ix, y; z)|^2 \geq h(t_0) = \rho^2 \]

Define \( \Omega = w : |w| < \rho \) then \( \phi(P(z), zP(z); z) \in \Omega \) for all real \( x \) and \( y = -n(1 + x^2)/2, z \in \Delta \). Therefore by application of Lemma 1 the result follows On taking \( \beta = 0 \) and \( \mu = 1, \delta = 1, n = 1, p = 1 \) in Theorem 1, we obtain the following interesting result.
Corollary 3 If \( f(z) \in A_1(p,0) = A_1(p) \) satisfies
\[
\left| \left( \frac{zf'(z)}{g(z)} - p \right) \left( \frac{zf'(z)}{g(z)} + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right) \right| \leq \rho, (z \in \Delta)
\]
where
\[
\rho^2 = \frac{(1 + t_0)}{t_0} \left[ \frac{9}{4} t_0^2 + \frac{5}{2} t_0 + \frac{1}{4} \right]
\]
t_0 is the smallest positive root by the equation
\[
3t^3 + \frac{19}{4} t^2 - \frac{1}{4} = 0
\]
then \( f(z) \in K_1(p,0) \).

Remark 1 On taking \( \beta = 0 \) and \( \mu = 0, \delta = 1, n = 1, p = 1 \) in Theorem ,
we obtain a known result due to Ravichandran et. Al. [1].

References


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