New sufficient conditions for univalence

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Abstract

The object of the present paper is to obtain new sufficient conditions on \( f''(z) \) which lead to some subclasses of univalent functions defined in the open unit disk.

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1 Introduction

Let \( A_n \) denote the class of functions of the form:

\[
(1) \quad f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots,
\]
which are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \). Let \( A_1 = A \), then a function \( f(z) \in A \) is said to be in the class \( S^*(\alpha) \), the class of starlike functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), if and only if

\[
(2) \quad \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U).
\]

Then \( S^* = S^*(0) \) is the class of starlike functions in \( U \). Further, if \( f(z) \in A \) satisfies

\[
(3) \quad \left| \frac{zf'(z)}{f(z)} \right| < \frac{\alpha \pi}{2}, \quad (z \in U)
\]

for some \( 0 < \alpha \leq 1 \), then \( f(z) \) said to be strongly starlike function of order \( \alpha \) in \( U \), and this class denoted by \( \overline{S}^*(\alpha) \). Note that \( \overline{S}^*(1) = S^* \).

Furthermore, let \( \mathcal{K}(\alpha) \), \( 0 \leq \alpha < 1 \), which consists of functions \( f(z) \in A \) such that

\[
(4) \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U),
\]

and \( \mathcal{K} = \mathcal{K}(0) \) is the class of convex functions in \( U \). Also, if \( f(z) \in A \) satisfies

\[
(5) \quad \left| \frac{zf''(z)}{f'(z)} \right| < \frac{\alpha \pi}{2}, \quad (z \in U)
\]

for some \( 0 < \alpha \leq 1 \), then \( f(z) \) said to be strongly convex function of order \( \alpha \) in \( U \), and this class denoted by \( \overline{\mathcal{K}}(\alpha) \). Note that \( \overline{\mathcal{K}}(1) = \mathcal{K} \) and if \( zf''(z) \in \overline{S}^*(\alpha) \) then \( f(z) \in \overline{\mathcal{K}}(\alpha) \).

All the above mentioned classes are subclasses of univalent fictions in \( U \).

There are many results for sufficient conditions of functions \( f(z) \) which are analytic in \( U \) to be starlike, convex, strongly starlike and strongly convex
functions have been given by several researchers (see [9],[1],[2],[7],[6],[5]). In this paper we will study \( \lambda \) such that the conditions \( |f''(z)| \leq \lambda, z \in U \), implies that \( f(z) \) belongs to one of the classes defined above.

In order to prove our main results, we shall need the following lemmas.

**Lemma 1** ([9]) If \( f(z) \in A \) satisfies

\[
|f'(z) - 1| < 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} \quad (z \in U)
\]

where \( a = \sin(\alpha \pi/2) \), \( 0 < \alpha \leq 1 \), then \( f(z) \in S^*(\alpha) \).

**Lemma 2** ([4]) If \( f(z) \in A_n \), satisfies

\[
|f'(z) - 1| < \frac{(1 - \alpha)(n + 1)}{\alpha + \sqrt{(n + 1)^2 + 1}}, \quad (z \in U)
\]

where \( 0 \leq \alpha < 1 \), then \( f(z) \in S^*(\alpha) \).

**Lemma 3** ([8]) If \( f(z) \in A_n \), satisfies

\[
|f'(z) + \alpha zf''(z) - 1| < \frac{(\alpha - 2)(n\alpha + 1)}{\alpha(n + 1)} \quad (z \in U)
\]

where \( \alpha > 2 \), then \( f(z) \in K \).

## 2 Main Results

Employing the same method used by Nunokawa et al.[6], we prove the following
Theorem 1  If $f(z) \in A$ satisfies

\[ |f''(z)| \leq 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} \quad (z \in U; \ 0 < \alpha \leq 1) \]

where $a = \sin(\alpha \pi/2)$, then $f(z) \in \mathcal{S}^*(\alpha)$.

**Proof.** Noting that

\[
|f'(z) - 1| = \left| \int_0^z f''(\sigma)d\sigma \right| \leq \int_0^{|z|} |f''(te^{i\theta})| dt \leq 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} \int_0^{|z|} dt
\]

\[
= 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} |z| < 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}}
\]

Hence, by Lemma 1, we conclude that $f(z) \in \mathcal{S}^*(\alpha)$.

**Corollary 1** Let $f(z) \in A, z \in U$ then

(i) $|f''(z)| \leq 2\sqrt{5}/5 = 0.8944\ldots$ implies $f(z) \in \mathcal{S}^*$;

(ii) $|f''(z)| \leq \sqrt{(5 - 2\sqrt{3})/13} = 0.3437\ldots$ implies $f(z) \in \mathcal{S}^*(1/3)$;

(iii) $|f''(z)| \leq \sqrt{(10 - 4\sqrt{2})/17} = 0.5054\ldots$ implies $f(z) \in \mathcal{S}^*(1/2)$;

and

(iii) $|f''(z)| \leq \sqrt{21}/7 = 0.6546\ldots$ implies $f(z) \in \mathcal{S}^*(2/3)$.

**Remark 1** The result from Corollary 1 (i) was obtained by Nunokawa et al.[6].

Now, we derive
Theorem 2 If \( f(z) \in A \) satisfies
\[
|f''(z)| \leq a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} \quad (z \in \mathcal{U}; \ 0 < \alpha \leq 1)
\]
where \( a = \sin(\alpha \pi/2) \), then \( f(z) \in \mathcal{K}(\alpha) \).

Proof. It follows that
\[
|zf''(z)' - 1| = |f'(z) + zf''(z) - 1| \leq |f'(z) - 1| + |zf''(z)|
\]
\[
\leq \left| \int_0^z f''(t)dt \right| + |zf''(z)| \leq \int_0^{|z|} |f''(t)dt| + a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} |z|
\]
\[
\leq 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}} |z| < 2a \sqrt{\frac{5 - 4\sqrt{1 - a^2}}{16a^2 + 9}}
\]

Therefore, using Lemma 1, we see that \( zf'(z) \in \mathcal{S}^{*}(\alpha) \), or \( f(z) \in \mathcal{K}(\alpha) \).

Corollary 2 Let \( f(z) \in A, z \in \mathcal{U} \) then

(i) \( |f''(z)| \leq \sqrt{5}/5 = 0.4472 \ldots \) implies \( f(z) \in \mathcal{K} \);

(ii) \( |f''(z)| \leq \frac{1}{2} \sqrt{(5 - 2\sqrt{3})/13} = 0.1718 \ldots \) implies \( f(z) \in \mathcal{K}(1/3) \);

(iii) \( |f''(z)| \leq \frac{1}{2} \sqrt{(10 - 4\sqrt{2})/17} = 0.2527 \ldots \) implies \( f(z) \in \mathcal{K}(1/2) \);

and

(iii) \( |f''(z)| \leq \frac{1}{2} \sqrt{21}/7 = 0.3273 \ldots \) implies \( f(z) \in \mathcal{K}(2/3) \).

Remark 2 The result from Corollary 2 (i) was obtained by Nunokawa et al.[6].

Applying the same method as in the proof of Theorem 1 and using Lemmas 2 instead of Lemma 1, we obtain the following theorem
**Theorem 3** If \( f(z) \in A_n \) satisfies
\[
|f''(z)| \leq \frac{(1 - \alpha)(n + 1)}{\alpha + \sqrt{(n + 1)^2 + 1}} \quad (z \in U; \ 0 \leq \alpha < 1)
\]
then \( f(z) \in S^*(\alpha) \).

Letting \( \alpha = 0 \) in Theorem 3, we obtain

**Corollary 3** If \( f(z) \in A_n \) satisfies
\[
|f''(z)| \leq \frac{n + 1}{\sqrt{(n + 1)^2 + 1}} \quad (z \in U)
\]
then \( f(z) \in S^* \).

**Remark 3** Letting \( n = 1 \) in Corollary 3, we obtain the result (i) from Corollary 2.5 which was obtained by Nunokawa et al.[6].

Finally, we prove

**Theorem 4** If \( f(z) \in A_n \), satisfies
\[
|f''(z)| < \frac{(\alpha - 2)(n\alpha + 1)}{\alpha(\alpha + 1)(n + 1)} \quad (z \in U)
\]
where \( \alpha > 2 \), then \( f(z) \in K \).

**Proof.** It follows that
\[
|f'(z) + \alpha zf''(z) - 1| \leq |f'(z) - 1| + \alpha |zf''(z)|
\]
\[
\leq \left| \int_0^z f''(t)dt \right| + \alpha |zf''(z)| \leq \int_0^{|z|} |f''(t)dt| + \frac{(\alpha - 2)(n\alpha + 1)}{\alpha(\alpha + 1)(n + 1)} |z|
\]
\[
\leq \frac{(\alpha - 2)(n\alpha + 1)}{\alpha(n + 1)} |z| < \frac{(\alpha - 2)(n\alpha + 1)}{\alpha(n + 1)},
\]
using Lemma 3, we have \( f(z) \in K \).

Letting \( \alpha \to \infty \) in Theorem 4, we have the following result obtained by Mocanu[3].
Corollary 4 If $f(z) \in A_n$, satisfies

\begin{equation}
|f''(z)| < \frac{n}{n+1} \quad (z \in \mathcal{U})
\end{equation}

then $f(z) \in \mathcal{K}$.

Letting $n = 1$ in Corollary 4, we have the following result obtained by Obradović[7].

Corollary 5 If $f(z) \in A$, satisfies

\begin{equation}
|f''(z)| < \frac{1}{2} \quad (z \in \mathcal{U})
\end{equation}

then $f(z) \in \mathcal{K}$.

References


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