Relation between Greek means and various means \(^1\)

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Abstract

In this paper, we obtain some inequalities between Greek means and various means. Further, we deduced the best possible values of various means with \(G_{n_{\mu, r}}(a, b)\) and \(g_{n_{\mu, r}}(a, b)\). Also we studied the partial derivatives of important means and the value of \(\alpha\) of second order partial derivatives.

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1 Introduction

In ([1]), Ten Greek means are defined on the basis of proportions of which six means are named and four means are unnamed and some distinguished results are obtained. In ([5], [6]) authors defined Oscillatory, $r^{th}$ Oscillatory means and its duals and obtained some new inequalities and the best possible values with Logarithmic mean, Identric mean and Power mean. In ([7]) authors defined $Gn_{\mu,r}(a, b)$, $gn_{\mu,r}(a, b)$ deduced some important results and also shown applications to Ky-Fan inequalities. Here we find the best possible values of the parameters $\mu$, $r$ for which $F_4$, $F_5$ and $F_6$ are satisfied by the inequalities (15) to (22). Further in ([1]), the partial derivatives of means and some related results are given, using which we obtained parameter $\alpha$ for various means.

Let $a, b > 0$, then

(1) $A(a, b) = F_1(a, b) = \frac{a + b}{2}$

(2) $G(a, b) = F_2(a, b) = \sqrt{ab}$

(3) $F_3(a, b) = \frac{2ab}{a + b}$

(4) $C(a, b) = F_4(a, b) = \frac{a^2 + b^2}{a + b}$

(5) $F_5(a, b) = \frac{a - b + \sqrt{(a - b)^2 + 4b^2}}{2}$

(6) $F_6(a, b) = \frac{b - a + \sqrt{(a - b)^2 + 4a^2}}{2}$
are respectively called Arithmetic mean, Geometric mean, Harmonic mean, contra Harmonic mean, first contra Geometric mean, second contra Geometric mean. Above are called named six Greek means.

\[(7) \quad L(a, b) = \begin{cases} \frac{a - b}{\ln a - \ln b} & a \neq b \\ \frac{a}{b} & a = b \end{cases}\]

\[(8) \quad I(a, b) = \begin{cases} e^{\left(\frac{a \ln a - b \ln b}{a - b}\right) - 1} & a \neq b \\ \frac{a}{b} & a = b \end{cases}\]

\[(9) \quad M_r(a, b) = \begin{cases} \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} & r \neq 0 \\ \sqrt{ab} & r = 0 \end{cases}\]

\[(10) \quad H(a, b) = \frac{a + \sqrt{ab} + b}{3}\]

are respectively called Logarithmic mean, Identric mean and Power mean and Heron mean.

**Definition 1 ([7])** For positive numbers \(a\) and \(b\), \(r\) be a positive real number and \(\mu \in (-2, \infty)\). Then \(G_{n+r}(a, b)\) and \(g_{n+r}(a, b)\) are defined as

\[(11) \quad G_{n+r}(a, b) = \begin{cases} \frac{2}{\mu+2} A(a, b) + \frac{\mu}{\mu+2} M_r(a, b) & r \neq 0 \\ \frac{2}{\mu+2} A(a, b) + \frac{\mu}{\mu+2} G(a, b) & r = 0 \end{cases}\]

and

\[(12) \quad g_{n+r}(a, b) = \begin{cases} M_{\frac{\mu}{\mu+2}}(a, b) A_{\frac{\mu}{\mu+2}}(a, b) & r \neq 0 \\ G_{\frac{\mu}{\mu+2}}(a, b) A_{\frac{\mu}{\mu+2}}(a, b) & r = 0 \end{cases}\].
\textbf{Definition 2} ([6]) Let $\alpha \in [0,1]$ and $r \geq 0$, then $r^{th}$ Oscillatory mean and its dual are defined by
\begin{equation}
O = O(a, b; \alpha, r) = \alpha M_r(a, b) + (1 - \alpha) A(a, b)
\end{equation}
and
\begin{equation}
o = o(a, b; \alpha, r) = M_r^\alpha(a, b)A^{1-\alpha}(a, b).
\end{equation}

Let us conclude the introduction by a brief description of the contents of the paper. Section 2 contains new inequalities involving Greek means and other means and its proof are given. Also, we present table 1 contain the best possible value of important means with $Gn_{\mu,r}(a, b)$ and $gn_{\mu,r}(a, b)$, power mean, Oscillatory mean and $r^{th}$ Oscillatory mean. Finally, Section 3 contains partial derivatives and consequences of symmetric mean, $\alpha$ values for important means are tabulated in Table 2 and two remarks.

\section{Some Inequalities}

\textbf{Theorem 1} For $\mu_1, \mu_2 \neq -2, r \neq 0, 3$ and if $\mu_1 \leq \frac{4}{r-3} \leq \mu_2$, then
\begin{equation}
(i) \quad gn_{\mu_2,r}(a, b) \leq F_4(a, b) \leq Gn_{\mu_1,r}(a, b).
\end{equation}
Furthermore $\mu_1 = \mu_2 = -\frac{4}{r-3}$ is the best possible for (15).
\begin{equation}
(ii) \quad gn_{\mu_2,0}(a, b) \leq F_4(a, b) \leq Gn_{\mu_1,0}(a, b).
\end{equation}
Furthermore $\mu_1 = \mu_2 = -\frac{4}{3}$ is the best possible for (16).
Proof. Applying Taylor’s theorem and by setting \( a = x = t + 1 \) and \( b = 1 \), we have
\[
F_4(x, 1) = F_4(t + 1, 1) = 1 + \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} - ... \\
Gn_{\mu_1,r}(x, 1) = Gn_{\mu_1,r}(t + 1, 1) = 1 + \frac{t}{2} - \frac{(1 - r)\mu_1 t^2}{(\mu_1 + 2)} + ... \\
gn_{\mu_2,r}(x, 1) = gn_{\mu_2,r}(t + 1, 1) = 1 + \frac{t}{2} - \frac{(1 - r)\mu_2 t^2}{(\mu_2 + 2)} + ...
\]
Consider \( gn_{\mu_2,r}(a, b) \leq F_4(a, b) \leq Gn_{\mu_1,r}(a, b) - \frac{(1 - r)\mu_2}{(\mu_2 + 2)} \leq \frac{1}{4} \leq \frac{(1 - r)\mu_1}{(\mu_1 + 2)} \)
with simple manipulation we have \( \mu_1 \leq \frac{4}{r} \leq \mu_2 \). Hence the proof of (15) and (16).

**Theorem 2** For \( \mu_1, \mu_2 \neq -2, r \neq 0, 2 \) and if \( \mu_1 \leq \frac{2}{r - 2} \leq \mu_2 \), then
\[
(i) \quad gn_{\mu_2,r}(a, b) \leq F_5(a, b) \leq Gn_{\mu_1,r}(a, b). \\

Furthermore \( \mu_1 = \mu_2 = \frac{2}{r - 2} \) is the best possible for (17)
\[
(ii) \quad gn_{\mu_2,0}(a, b) \leq F_5(a, b) \leq Gn_{\mu_1,0}(a, b). \\

Furthermore \( \mu_1 = \mu_2 = -1 \) is the best possible for (18).

**Corollary 1** For \( \mu_1, \mu_2 \neq -2, r \neq 0, 2 \) and if \( \mu_1 \leq \frac{2}{r - 2} \leq \mu_2 \), then
\[
(i) \quad gn_{\mu_2,r}(a, b) \leq F_6(a, b) = F_5(a, b) \leq Gn_{\mu_1,r}(a, b). \\

Furthermore \( \mu_1 = \mu_2 = \frac{2}{r - 2} \) is the best possible for (19).
\[
(ii) \quad gn_{\mu_2,0}(a, b) \leq F_6(a, b) = F_5(a, b) \leq Gn_{\mu_1,0}(a, b). \\

Furthermore \( \mu_1 = \mu_2 = -1 \) is the best possible for (20).
Theorem 3 Let $\alpha_1, \alpha_2 \in [0, 1]$, $r \neq 0, 1$, if $\alpha_1 \leq \frac{2}{r-1} \leq \alpha_2$, then

\begin{equation}
O(a,b; \alpha_1, r) \geq F_4(a,b) \geq o(a,b; \alpha_2, r).
\end{equation}

Furthermore $\alpha_1 = \alpha_2 = \frac{2}{r-1}$ is the best possible for (21).

Proof. Applying Taylor’s theorem and by setting $a = x = t + 1$ and $b = 1$, we have

\[ F_4(x, 1) = F_4(t + 1, 1) = 1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} - ... \]

\[ O(a,b; \alpha, r) = 1 + \frac{t}{2} - \frac{\alpha (1-r)}{8} t^2 + ... \]

\[ o(a,b; \alpha, r) = 1 + \frac{t}{2} - \frac{\alpha (1-r)}{8} t^2 + .... \]

Consider $\alpha_1 \leq \frac{2}{r-1} \leq \alpha_2$. With simple manipulations we get

\[ -\frac{\alpha_1 (1-r)}{8} \geq \frac{1}{4} \geq -\frac{\alpha_2 (1-r)}{8} \]

\[ 1 + \frac{t}{2} - \frac{\alpha_1 (1-r)}{8} t^2 + ... \geq 1 + \frac{t}{2} + \frac{1}{4} t^2 + ... \geq 1 + \frac{t}{2} - \frac{\alpha_2 (1-r)}{8} t^2 + ... \]

\[ O(a,b; \alpha_1, r) \geq F_4(a,b) \geq o(a,b; \alpha_2, r). \]

Furthermore $\alpha_1 = \alpha_2 = \frac{2}{r-1}$ is the best possible (for 21).

Theorem 4 Let $\alpha_1, \alpha_2 \in [0, 1]$, $r \neq 0, 1$, if $\alpha_1 \leq \frac{1}{r-1} \leq \alpha_2$, then

\begin{equation}
O(a,b; \alpha_1, r) \geq F_5(a,b) \geq o(a,b; \alpha_2, r).
\end{equation}

Furthermore $\alpha_1 = \alpha_2 = \frac{1}{r-1}$ is the best possible for (22).

Corollary 2 Let $\alpha_1, \alpha_2 \in [0, 1]$, $r \neq 0, 1$, if $\alpha_1 \leq \frac{1}{r-1} \leq \alpha_2$, then

\begin{equation}
O(a,b; \alpha_1, r) \geq F_5(a,b) = F_6(a,b) \geq o(a,b; \alpha_2, r).
\end{equation}

Furthermore $\alpha_1 = \alpha_2 = \frac{1}{r-1}$ is the best possible for (23).
The proofs of the following remarks are obvious.

**Remark 1**

\[ F_3 \leq F_2 \leq L \leq M_{1/3} \leq M_{1/2} \leq H \leq M_{2/3} \leq F_1 \leq F_6 \leq F_5 \leq F_4. \]

**Remark 2**

\[ F_3 \leq F_2 \leq L \leq I \leq F_1 \leq F_6 \leq F_5 \leq F_4. \]

The following table gives the best possible value of important means with \( G_{\mu,r}(a, b) \) and \( gn_{\mu_2,r}(a, b) \) power mean, oscillatory mean and \( r \)th oscillatory mean.

<table>
<thead>
<tr>
<th>Important means</th>
<th>( G_{\mu,0}(a, b) )</th>
<th>( G_{\mu,r}(a, b) )</th>
<th>( O(a, b; a, r) )</th>
<th>( O(a, b, a) )</th>
<th>( M_r(a, b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic mean</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Geometric mean</td>
<td>( \infty )</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{1}{1-r} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Contra Harmonic mean</td>
<td>( -\frac{4}{3} )</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{2}{1-r} )</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>I Contra Geometric mean</td>
<td>( -1 )</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{1}{1-r} )</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>II Contra Geometric mean</td>
<td>( -1 )</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{1}{1-r} )</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>Logarithmic mean</td>
<td>4</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{r}{3} )</td>
<td>( \frac{r}{3} )</td>
</tr>
<tr>
<td>Identric Mean</td>
<td>1</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{1}{1-r} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>Heron Mean</td>
<td>1</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{1}{1-r} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>Power Mean</td>
<td>( -1 )</td>
<td>( \frac{2}{1-r} )</td>
<td>( \frac{1}{1-r} )</td>
<td>( 1 - r )</td>
<td>( - )</td>
</tr>
</tbody>
</table>

### 3 Partial Derivatives and Consequences

For a symmetric mean \( M(a, b) \) the partial derivatives are exist, then we have

\[ M_a(c, c) + M_b(c, c) = 1 \]

(24)
(25) \[ M_a(c,c) \geq 0 \text{ and } M_b(c,c) \geq 0 \]

(26) \[ 0 \leq M_a(c,c) \leq 1 \text{ and } 0 \leq M_b(c,c) \leq 1 \]

(26) property does not hold for arbitrary point.

(27) \[ M_a(c,c) = M_b(c,c) = \frac{1}{2} \]

(28) \[ M_{aa}(c,c) + 2M_{ab}(c,c) + M_{bb}(c,c) = 0 \]

(29) \[ M_{aa}(c,c) = -M_{ab}(c,c) = M_{bb}(c,c) \]

(30) \[ M_{aa}(c,c) = \frac{\alpha}{c} \text{ where } \alpha \in \mathbb{R}. \]

The proofs of the above results are obtained by simple direct computations.

Table 2

<table>
<thead>
<tr>
<th>Important means</th>
<th>Notation</th>
<th>The value of ‘(\alpha)’</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic Mean</td>
<td>(F_1(a,b))</td>
<td>0</td>
</tr>
<tr>
<td>Geometric mean</td>
<td>(F_2(a,b))</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>Harmonic Mean</td>
<td>(F_3(a,b))</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>Logarithmic Mean</td>
<td>(L(a,b))</td>
<td>(\frac{1}{e})</td>
</tr>
<tr>
<td>Heron Mean</td>
<td>(h(a,b))</td>
<td>(\frac{1}{\sqrt{2}})</td>
</tr>
<tr>
<td>Identric Mean</td>
<td>(I(a,b))</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>Power Mean</td>
<td>(M_r(a,b))</td>
<td>(\frac{1}{r^2})</td>
</tr>
<tr>
<td>Contra Harmonic mean</td>
<td>(F_4(a,b))</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>First Contra Geometric mean</td>
<td>(F_5(a,b))</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>Second Contra Geometric mean</td>
<td>(F_6(a,b))</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>Oscillatory mean</td>
<td>(o(a,b;\alpha))</td>
<td>(\frac{\alpha}{\alpha + 2})</td>
</tr>
<tr>
<td>(r^{th}) Oscillatory mean</td>
<td>(o(a,b;\alpha))</td>
<td>(\frac{-\alpha(1-x)}{\alpha(1-x) + 2})</td>
</tr>
<tr>
<td>Definition 1.</td>
<td>(G_{n_\mu,r}(a,b)) and (g_{n_\mu,r}(a,b)) ((r = 0))</td>
<td>(\frac{-\mu(1-x)}{4(\mu + 2)})</td>
</tr>
<tr>
<td>Definition 1.</td>
<td>(G_{n_\mu,r}(a,b)) and (g_{n_\mu,r}(a,b)) ((r \neq 0))</td>
<td>(\frac{-\mu}{4\mu + 2})</td>
</tr>
</tbody>
</table>
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References


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