\textbf{q-Extensions for the Apostol-Genocchi Polynomials}$^1$

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Abstract

In this paper, we define the Apostol-Genocchi polynomials and \( q \)-Apostol-Genocchi polynomials. We give the generating function and some basic properties of \( q \)-Apostol-Genocchi polynomials. Several interesting relationships are also obtained.

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1 Introduction, definitions and motivation

Throughout this paper, we always make use of the following notation: \( \mathbb{N} = \{1, 2, 3, \ldots\} \) denotes the set of natural numbers, \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \) denotes the set of nonnegative integers, \( \mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\} \) denotes the set of

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The usual binomial theorem is \( \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!} \) \((n \in \mathbb{N})\); the rising factorial is \( f^{(n)} = n(n+1) \cdots (n+k-1) \); the \( q \)-shifted factorial is \( (a;q)_0 = 1, (a;q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1}), k = 1, 2, \ldots; (a;q)_\infty = (1-a)(1-aq) \cdots (1-aq^k) \cdots = \prod_{k=0}^{\infty} (1-aq^k) \), \(|q| < 1; \ a, q \in \mathbb{C}\).

Clearly, \( (a;q)_k = \frac{(aq)_\infty}{(a)_\infty} \).

The \( q \)-number or \( q \)-basic number is defined by \([a]_q = \frac{1 - q^a}{1 - q}, q \neq 1, \ |q| < 1; \ a, q \in \mathbb{C}\); the \( q \)-numbers factorial is defined by \([n]_q! = [1]_q[2]_q \cdots [n]_q, \ (n \in \mathbb{N})\). The \( q \)-numbers shifted factorial is defined by \(([a]_q)_n = [a]_{qn} = [a][a+1]_q \cdots [a+n-1]_q \) \((n \in \mathbb{N}, a \in \mathbb{C})\). Clearly, \( \lim_{q \to 1}[a]_q = a, \ \lim_{q \to 1}[n]_q! = n!, \ \lim_{q \to 1}([a]_q)_n = (a)_n \).

The usual binomial theorem
\[
\frac{1}{(1-z)^\alpha} = \sum_{n=0}^{\infty} \binom{-\alpha}{n} (-z)^n := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n, \quad (z, \alpha \in \mathbb{C}; \ |z| < 1).
\]

The \( q \)-binomial theorem
\[
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(a z; q)_\infty}{(z; q)_\infty}, \quad (z, q \in \mathbb{C}; \ |z| < 1, |q| < 1).
\]

A special case of (1.2), for \( a = q^\alpha (\alpha \in \mathbb{C}) \), can be written as follows:
\[
\frac{1}{(z; q)_\alpha} = \frac{(q^\alpha z; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(q^\alpha)_n}{(q)_n} z^n := \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} z^n,
\]
(z, q, \alpha \in \mathbb{C}; \ |z| < 1, |q| < 1).

The above \( q \)-standard notation can be found in [2].

The Genocchi numbers \( G_n \) and polynomials \( G_n(x) \) together with their generalizations \( G_n^{(\alpha)}(x) \) \((\alpha \in \mathbb{R} \oplus \mathbb{C})\), are usually defined by means of the following generating functions (see [5] p. 532-533):
\[
\left( \frac{2z}{e^z + 1} \right)^\alpha = \sum_{n=0}^{\infty} G_n^{(\alpha)} z^n \quad (|z| < \pi),
\]
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\[(1.5) \quad \left(\frac{2z}{e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi).\]

Obviously, for \(\alpha = 1\), Genocchi polynomials \(G_n(x)\) and numbers \(G_n\) are

\[(1.6) \quad G_n(x) := G_n^{(1)}(x) \quad \text{and} \quad G_n := G_n(0) \quad (n \in \mathbb{N}_0),\]

respectively.

We now introduce the following extensions of Genocchi polynomials of higher order based on the idea of Apostol (see, for details, [1]).

**Definition 1.1.** The Apostol-Genocchi numbers and polynomials of order \(\alpha\) are respectively defined by means of the generating functions:

\[(1.7) \quad \left(\frac{2z}{\lambda e^z + 1}\right)^\alpha = \sum_{n=0}^{\infty} G_n^{(\alpha)}(\lambda) \frac{z^n}{n!} \quad (|z| < |\log(-\lambda)|),\]

\[(1.8) \quad e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!} \quad (|z| < |\log(-\lambda)|).\]

Clearly, we have

\[(1.9) \quad G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1), \quad G_n^{(\alpha)}(\lambda) := G_n^{(\alpha)}(0; \lambda),\]

\[G_n(x; \lambda) := G_n^{(1)}(x; \lambda) \quad \text{and} \quad G_n(\lambda) := G_n^{(1)}(\lambda),\]

where \(G_n(\lambda), G_n^{(\alpha)}(\lambda)\) and \(G_n(x; \lambda)\) denote the so-called Apostol-Genocchi numbers, Apostol-Genocchi numbers of order \(\alpha\) and Apostol-Genocchi polynomials respectively.

It follows that we give the following \(q\)-extensions for Apostol-Genocchi polynomials of order \(\alpha\).

**Definition 1.2.** The \(q\)-Apostol-Genocchi numbers and polynomials of order \(\alpha\) are respectively defined by means of the generating functions:

\[(1.10) \quad W_n^{(\alpha)}(t) = (2t)^\alpha \sum_{n=0}^{\infty} \frac{[\alpha]_q^n}{[n]_q!} (-\lambda)^n q^n e^{[\lambda]_q t} \sum_{n=0}^{\infty} G_n^{(\alpha)}(\lambda) \frac{t^n}{n!}, \quad (q, \alpha, \lambda \in \mathbb{C}; \ |q| < 1).\]
\begin{equation}
W^{(\alpha)}_{x,\lambda;q}(t) = (2t)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{[n+x]_q t} \\
= \sum_{n=0}^{\infty} G^{(\alpha)}_{n;q}(x;\lambda) \frac{t^n}{n!}; \quad (q, \alpha, \lambda \in \mathbb{C}; \ |q| < 1).
\end{equation}

Obviously,

\[\lim_{q \to 1} G^{(\alpha)}_{n;q}(x;\lambda) = G^{(\alpha)}_{n}(x;\lambda), \quad \lim_{q \to 1} G^{(\alpha)}_{n}(\lambda) = G^{(\alpha)}_{n}(\lambda)\]

and

\[\lim_{q \to 1} G^{(\alpha)}_{n;q}(x) = G^{(\alpha)}_{n}(x), \quad \lim_{q \to 1} G^{(\alpha)}_{n;q} = G^{(\alpha)}.\]

We recall that a family of the Hurwitz-Lerch Zeta function \(\Phi^{(\rho,\sigma)}_{\mu,\nu}(z, s, a)\) [4, p. 727, Eq. (8)] is defined by

\begin{equation}
\Phi^{(\rho,\sigma)}_{\mu,\nu}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n + a)^s},
\end{equation}

\((\mu \in \mathbb{C}; \ a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \rho, \sigma \in \mathbb{R}^+; \ \rho < \sigma \text{ when } s, z \in \mathbb{C}; \ \rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < 1; \ \rho = \sigma \text{ and } \Re(s - \mu + \nu) > 1 \text{ when } |z| = 1),\)

contains, as its \textit{special} cases, not only the Hurwitz-Lerch Zeta function

\begin{equation}
\Phi^{(\sigma,\sigma)}_{\nu,\nu}(z, s, a) = \Phi^{(0,0)}_{\mu,\nu}(z, s, a) = \Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s},
\end{equation}

but also the following generalized Hurwitz-Zeta function introduced and studied earlier by Goyal and Laddha [3 p. 100, Eq. (1.5)]

\begin{equation}
\Phi^{(1,1)}_{\mu,1}(z, s, a) = \Phi_{\mu}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n + a)^s},
\end{equation}

which, for convenience, are called the \textit{Goyal-Laddha-Hurwitz-Lerch Zeta function}.

It follows that we introduce the following definitions.
**Definition 1.3.** The \(q\)-Goyal-Laddha-Hurwitz-Lerch Zeta function is defined by

\[
(1.15) \quad \Phi_{\mu,q}(z, s, a) := \sum_{n=0}^{\infty} \frac{([\mu]_q)_n}{[n]_q!} \frac{z^n q^{n+a}}{[n+a]_q^s}, \quad (\mu, s \in \mathbb{C}; \ \Re(a) > 0; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-).
\]

Setting \(\mu = 1\) in (1.15), we have

**Definition 1.4.** The \(q\)-Hurwitz-Lerch Zeta function is defined by

\[
(1.16) \quad \Phi_q(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n q^{n+a}}{[n+a]_q^s}, \quad (s \in \mathbb{C}; \ \Re(a) > 0; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-).
\]

The aim of this paper is to give another generating function of \(q\)-Apostol-Genocchi polynomials. Some basic properties are also studied. We obtain several interesting relationships between these polynomials and the generalized Zeta functions.

## 2 Generating functions of the \(q\)-Apostol-Genocchi polynomials of higher order

By (1.3) and (1.11), yields

\[
(2.1) \quad W^{(\alpha)}_{x;\lambda,q}(t) = (2t)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{[n+x]_q} t^n
\]

\[= (2t)^\alpha e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{-\frac{x+n}{1-q} t^n}
\]

\[= (2t)^\alpha e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(1-q)^k} \frac{t^k}{k!} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda q^{k+1})^n
\]

\[= (2t)^\alpha e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1}; q)_\alpha} \left( \frac{1}{1-q} \right)^k \frac{t^k}{k!}.
\]
Therefor, we obtain the generating function of $G^{(\alpha)}_{n;q}(x; \lambda)$ as follows:

(2.2) 
\[
W_{x;\lambda;q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1}; q)_\alpha} \left( \frac{1}{1-q} \right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} G^{(\alpha)}_{n;q}(x; \lambda) \frac{t^n}{n!}.
\]

Clearly,

(2.3) 
\[
W_{\lambda;q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(-\lambda q^{k+1}; q)_\alpha} \left( \frac{1}{1-q} \right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} G^{(\alpha)}_{n;q}(\lambda) \frac{t^n}{n!}.
\]

Setting $\lambda = 1$ in (2.2) and (2.3) respectively, we deduce the generating functions of $G^{(\alpha)}_{n;q}(x)$ and $G^{(\alpha)}_{n;q}$ as follows:

(2.4) 
\[
W_{x;q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-q^{k+1}; q)_\alpha} \left( \frac{1}{1-q} \right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} G^{(\alpha)}_{n;q}(x) \frac{t^n}{n!}.
\]

and

(2.5) 
\[
W_{q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(-q^{k+1}; q)_\alpha} \left( \frac{1}{1-q} \right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} G^{(\alpha)}_{n;q}(\lambda) \frac{t^n}{n!}.
\]

It follows that we derive readily the following formulas by (2.2) and (2.3) for $\alpha = \ell \in \mathbb{N}$.

(2.6) 
\[
G^{(\ell)}_{n;q}(\lambda) = \frac{2^\ell}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} (-1)^{k-\ell} \frac{k!}{(-\lambda q^{k+1}; q)_\ell}
\]

and

(2.7) 
\[
G^{(\ell)}_{n;q}(x; \lambda) = \frac{2^\ell}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} (-1)^{k-\ell} \frac{k!}{(-\lambda q^{k+1}; q)_\ell} x^{(k+1)}.
\]

Setting $\lambda = 1$ in (2.6) and (2.7) respectively, we deduce the explicit formulas as follows:

(2.8) 
\[
G^{(\ell)}_{n;q} = \frac{2^\ell}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} (-1)^{k-\ell} \frac{k!}{(-q^{k+1}; q)_\ell}
\]
and

\[
G_{n,q}^{(\ell)}(x) = \frac{2^\ell}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} (-1)^{k-\ell} \{k\}_q q^{(k-\ell+1)x} \left(-q^{k-\ell+1}; q\right)_\ell.
\]

3 Some properties of the \(q\)-Apostol-Genocchi polynomials of higher order

In this Section, we shall derive some basic properties of the \(q\)-Apostol-Genocchi polynomials.

**Proposition 3.1.** The special values for \(q\)-Apostol-Genocchi polynomials and numbers of higher order \((n, \ell \in \mathbb{N}; \alpha, \lambda \in \mathbb{C})\)

\[
G_{n,q}^{(\alpha)}(\lambda) = G_{n,q}^{(\alpha)}(0; \lambda), \quad G_{n,q}^{(0)}(x; \lambda) = q^x [x]_q^n;
\]

\[
G_{0,q}^{(\alpha)}(x; \lambda) = \delta_{\alpha,0}, \quad G_{n,q}^{(\ell)}(x; \lambda) = 0 \quad (0 \leq n \leq \ell - 1).
\]

\(\delta_{n,k}\) being the Kronecker symbol.

**Proposition 3.2.** The formula of \(q\)-Apostol-Genocchi polynomials of higher order in terms of \(q\)-Apostol-Genocchi numbers of higher order

\[
G_{n,q}^{(\alpha)}(x; \lambda) = \sum_{k=0}^{n} \binom{n}{k} G_{k,q}^{(\alpha)}(\lambda) q^{(k-\alpha+1)x} [x]_q^{n-k}.
\]

**Proof.** By (1.11) and (1.10), yields

\[
W_{x;\lambda,q}^{(\alpha)}(t) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x; \lambda) \frac{t^n}{n!} = (2t)^\alpha \sum_{n=0}^{\infty} \frac{[\alpha]_q^n}{[n]_q!} (-\lambda)^n q^{n+x} e^{[n+x]_q t}
\]

\[
= (2t)^\alpha q^x e^{[x]_q t} \sum_{n=0}^{\infty} \frac{[\alpha]_q^n}{[n]_q!} (-\lambda)^n q^{n+x} e^{[n]_q q^x t}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} G_{k,q}^{(\alpha)}(\lambda) q^{(k-\alpha+1)x} [x]_q^{n-k} \frac{t^n}{n!}.
\]

Comparing the coefficients of \(\frac{t^n}{n!}\) on both sides of (3.3), we lead immediately to the desired (3.2).
Proposition 3.3 (Difference equation).

\[(3.4) \quad \lambda q^{a-1} G_{n;q}^{(a)}(x + 1; \lambda) + G_{n+1;q}^{(a)}(x; \lambda) = 2n G_{n-1;q}^{(a-1)}(x; \lambda) \quad (n \geq 1).\]

Proof. It is easy to observe that

\[(3.5) \quad \sum_{n=0}^{\infty} \frac{[[\alpha - 1]_q]_n}{[n]_q !} (-\lambda)^n q^{n+x} e^{[n+x]_q t} = \lambda q^{a-1} \sum_{n=0}^{\infty} \frac{[[\alpha]_q]_n}{[n]_q !} (-\lambda)^n q^{n+x+1} e^{[n+x+1]_q t} + \sum_{n=0}^{\infty} \frac{[[\alpha]_q]_n}{[n]_q !} (-\lambda)^n q^{n+x} e^{[n+x]_q t}.\]

By (1.11) and (3.5), we obtain the desired (3.4).

Proposition 3.4 (Differential relationship).

\[(3.6) \quad \frac{\partial}{\partial x} G_{n;q}^{(a)}(x; \lambda) = G_{n+1;q}^{(a)}(x; \lambda) \log q + n \frac{\log q}{q - 1} q^n G_{n-1;q}^{(a)}(x; \lambda q).\]

Proof. By (2.7), it is not difficult.

Proposition 3.5 (Integral formula).

\[(3.7) \quad \int_a^b q^n G_{n;q}^{(a)}(x; \lambda q) \, dx = \frac{1 - q}{n + 1} \int_a^b G_{n+1;q}^{(a)}(x; \lambda) \, dx + \frac{q - 1}{\log q} \frac{G_{n+1;q}^{(a)}(b; \lambda) - G_{n+1;q}^{(a)}(a; \lambda)}{n + 1}.\]

Proof. It is easy to obtain (3.7) by (3.6).

Proposition 3.6 (Addition theorem).

\[(3.8) \quad G_{n;q}^{(a)}(x + y; \lambda) = \sum_{k=0}^{n} \binom{n}{k} G_{k;q}^{(a)}(x; \lambda) q^{(k+a+1)y} [y]_q^{n-k}.\]
Proof. By (1.11), yields

\begin{equation}
W_{x+y,\lambda,q}^{(\alpha)}(t) = \sum_{n=0}^{\infty} G_{n;q}^{(\alpha)}(x+y;\lambda) \frac{t^n}{n!} = (2t)^{\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_q^n}{[n]_q!} (-\lambda)^n q^{n+x+y} e^{[n+x+y]t}
\end{equation}

\begin{align*}
&= (2t)^{\alpha} q^y e^{yt} \sum_{n=0}^{\infty} \frac{[\alpha]_q^n}{[n]_q!} (-\lambda)^n q^{n+x} e^{[n+x]y q^n t} \\
&= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \binom{n}{k} G_{k;q}^{(\alpha)}(x;\lambda) q^{(k-\alpha+1)y} [y]_{1;q}^{n-k} \right] \frac{t^n}{n!}.
\end{align*}

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (3.9), we can arrive at formula (3.8) immediately.

**Proposition 3.7** (Theorem of complement).

\begin{align}
G_{n;q}^{(\alpha)}(\alpha - x;\lambda) &= \frac{(-1)^{n-\alpha}}{\lambda^n} q^{\alpha-(2)} - n G_{n;q-1}^{(\alpha)}(x;\lambda^{-1}), \\
G_{n;q}^{(\alpha)}(\alpha + x;\lambda) &= \frac{(-1)^{n-\alpha}}{\lambda^n} q^{\alpha-(2)} - n G_{n;q-1}^{(\alpha)}(-x;\lambda^{-1}).
\end{align}

**Proof.** It follows that by (2.7).

**Proposition 3.8** (Recursive formulas).

\begin{align}
(n-\alpha) G_{n;q}^{(\alpha)}(x;\lambda) &= n[x]_q G_{n-1;q}^{(\alpha)}(x;\lambda) - \frac{\lambda}{2} [\alpha]_q q^x G_{n;q}^{(\alpha+1)}(x+1;\lambda), \\
[\alpha]_q q^{x-\alpha} G_{n;q}^{(\alpha+1)}(x;\lambda) &= 2n ([\alpha]_q q^{x-\alpha} - [x]_q) G_{n-1;q}^{(\alpha)}(x;\lambda) + 2(n-\alpha) G_{n;q}^{(\alpha)}(x;\lambda).
\end{align}

**Proof.** We differentiate both side of (1.11) with respect to the variable $t$.}
yields

\begin{equation}
\frac{d}{dt}W_{x;\lambda q}(t) = \sum_{n=0}^{\infty} nG_{n,q}^{(\alpha)}(x;\lambda) \frac{t^{n-1}}{n!}
= 2\alpha(2t)^{\alpha-1} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{(n+x)q,t} + (2t)^{\alpha}[n+1]
\end{equation}

\begin{align*}
&+ x]_q \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{(n+x)q,t} \\
&= \alpha \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x;\lambda) \frac{t^{n-1}}{n!} + [x]_q \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x;\lambda) \frac{t^{n}}{n!} - \\
&- \frac{\lambda}{2}[\alpha]_q q^x \sum_{n=0}^{\infty} G_{n,q}^{(\alpha+1)}(x+1;\lambda) \frac{t^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty} \left[ \alpha G_{n,q}^{(\alpha)}(x;\lambda) + n[x]_q G_{n-1,q}^{(\alpha)}(x;\lambda) - \frac{\lambda}{2}[\alpha]_q q^x G_{n,q}^{(\alpha+1)}(x+1;\lambda) \right] \frac{t^{n-1}}{n!}.
\end{align*}

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (3.14), we get the desired (3.12).

We derive easily equation (3.13) by (3.4) and (3.12). The proof is complete.

**Remark 3.1.** When $q \to 1$, then the formulas in Proposition 3.1–Proposition 3.8 will become the corresponding formulas of Apostol-Genocchi polynomials of higher order. Further, letting $q \to 1, \alpha = 1$, then these formulas will become the corresponding formulas of Apostol-Genocchi polynomials.

**Remark 3.2.** When $\lambda = 1$, then the formulas in Proposition 3.1–Proposition 3.8 will become the corresponding formulas of $q$-Genocchi polynomials of higher order. Further, letting $\lambda = 1, \alpha = 1$, then these formulas will become the corresponding formulas of $q$-Genocchi polynomials.
4 Some explicit relationships between the $q$-Genocchi polynomials of higher order and $q$-Goyal-Laddha-Hurwitz-Lerch Zeta function

In this section, we give several interesting relationship between the Genocchi polynomials and Hurwitz-Lerch Zeta function.

We differentiate both side of (1.11) with respect to the variable $t$, for $\alpha = l \in \mathbb{N}$.

\begin{equation}
G_{n; q}^{(l)}(a; \lambda) = \frac{d^n}{d t^n} W_{a; \lambda; q}^{(l)}(t) \bigg|_{t=0} = 2^l \sum_{k=0}^{\infty} \frac{([l]_q)_k}{[k]_q!} (-\lambda)^k q^{k+a} \frac{d^n}{d t^n} \{ e^{[k+a]_q t} \} \bigg|_{t=0}
\end{equation}

we obtain the following theorem.

**Theorem 4.1.** The following relationship

\begin{equation}
G_{n; q}^{(l)}(a; \lambda) = 2^l \sum_{k=0}^{\infty} \frac{([l]_q)_k}{[k]_q!} (-\lambda)^k q^{k+a} [k + a]_q^{n-l} = 2^l \sum_{k=0}^{\infty} \frac{([l]_q)_k}{[k]_q!} [k + a]_q^{l-n},
\end{equation}

holds true between the $q$-Apostol-Genocchi polynomials of higher order and $q$-Goyal-Laddha-Hurwitz-Lerch Zeta function.

Taking $l = 1$ in (4.2), yields

**Corollary 4.1.** The following relationship

\begin{equation}
G_{n; q}(a; \lambda) = 2n \Phi_q(-\lambda, 1-n, a), \quad (n, l \in \mathbb{N}; \ n \geq l; \ |\lambda| \leq 1; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-),
\end{equation}

holds true between the $q$-Apostol-Genocchi polynomials and the $q$-Hurwitz-Lerch Zeta function.
Letting \( q \to 1 \) in (4.2), we have

**Corollary 4.2.** The following relationship

\[
G_n^{(l)}(a; \lambda) = 2^l \Phi_l(-\lambda, l-n, a), \quad (n, l \in \mathbb{N}; \ n \geq l; \ |\lambda| \leq 1; \ a \in \mathbb{C}\backslash\mathbb{Z}_0),
\]

holds true between the Apostol-Genocchi polynomials of higher order and Goyal-Laddha-Hurwitz-Lerch Zeta function.

Setting \( l = 1 \) in (4.4), we deduce the following interesting relationship

**Corollary 4.3.** The following relationship

\[
G_n(a; \lambda) = 2^n \Phi(-\lambda, 1-n, a), \quad (n \in \mathbb{N}; \ |\lambda| \leq 1; \ a \in \mathbb{C}\backslash\mathbb{Z}_0),
\]

holds true between the Apostol-Genocchi polynomials and Hurwitz-Lerch Zeta function.

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**References**


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