Mapping $\Phi^p$ in Normed Linear Spaces and Characterization of Orthogonality Problem of Best Approximations in 2-norm

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Abstract

In order to characterizations of best approximations have been given in 2-norm space $(X, \| . \| )$. Some generalization of the function $\Phi^p$ of Dragomir type have been given in the context where the said generalization help to formulate the characterizations what have been proposed in this article.

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1 Introduction

In a 2-normed linear space $(X, \| . \| )$ our present aim is to characterize the set of best approximations and related generalized orthogonality of a pair of elements in 2-normed space with reference to the 2-norm ([7] and [11]). We introduce below the $\Phi^p$ function and their properties as were done

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by Dragomir in an earlier reference [6]. We also study the boundedness, monotonicity and convexity properties of the generalized \( \Phi^p \) functions.

Let \((X, \|\cdot\|)\) be real 2-normed linear space. Consider the 2-norm derivative

\[
(y, x/z)_i = \lim_{t \to 0^-} \frac{\| x + ty, z \|^2 - \| x, z \|^2}{2t}
\]

and

\[
(y, x/z)_s = \lim_{t \to 0^+} \frac{\| x + ty, z \|^2 - \| x, z \|^2}{2t}
\]

which are well defined for every pair \(x, y \in X\) and \(z \in X/L\{x, y\}, G\) (where \(L(\{x,y\}, G)\) stands for the linear manifolds spaned by \(x\) and \(y\)).

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel([2],[3],[4],[5] and [6]), assuming that \(p, z \in \{s, i\}\) and \(p \neq 2\).

(i) \((x, x/z)_p = \| x, z \|^2\)

(ii) \((\alpha x, \beta y/z)_p = \alpha \beta (x, y/z)_p\) if \(\alpha, \beta \geq 0\)

(iii) \(|(x, y/z)_p| \leq \| x, z \| \cdot \| y, z \|

(iv) \((\alpha x + y, x/z)_p = \alpha (x, x/z)_p + (y, x/z)_p\) where \(\alpha \in R\)

(v) \((-x, y/z)_p = -(x, y/z)_q\)

(vi) \((x + y, w/z)_p \leq \| x, z \| \cdot \| w, z \| + (y, w/z)_p\)

(vii) The mapping \((\cdot, \cdot/z)_p\) is continuous and subadditive in the first variable for \(p = s\) or \(p = i\).

(viii) The element \(x \in X\) is Birkhoff orthogonal to the element \(y \in X\) (i.e. \(\| x + ty, z \| \geq \| x, z \| t\) for all \(t \in R\) and \(z \in X/L(\{x, y\}, G)\) if and only if

\[
(y, x/z)_i \leq 0 \leq (y, x/z)_s
\]

(ix) The 2-normed linear space \((X, \|\cdot\|)\) is smooth at the point \(x_0 \in X\setminus\{0\}\) if and only if the mapping \(y \to (y, x_0/z)_p\) is linear, or if and only if \((y, x_0/z)_s = (y, x_0/z)_i\) for all \(y \in X\) and \(z \in X/L(\{x, y\}, G)\). (x) If the 2-norm \(\|\cdot\|\) is induced by an 2-inner product \((\cdot, \cdot/z)\) then

\[
(y, x/z)_i = (y, x/z) = (y, x/z)_s\text{ for all }x, y \in X\text{ and }z \in X/L(\{x, y\}, G).
\]
2 Properties of the mapping $\Phi^p_{x,y/z}$

For three fixed linearly independent vectors $x, y$ in $X$ and $z \in X/L(\{x, y\}, G)$ we consider the mapping

$$\Phi^p_{x,y/z}(t) = \frac{(y, x + ty/z)_p}{\|x + ty, z\|}, \quad p = s \text{ or } p = i$$

which is well defined for all $t \in \mathbb{R}$.

**Theorem 2.1.** Let $(X, \|., .\|)$ be a real 2-normed linear space and $x, y, z$ two linearly independent vectors in $X$ and $z \in X/L(\{x, y\}, G)$. Then

(i) The mapping $\Phi^p_{x,y/z}$ is bounded on $\mathbb{R}$ with

$$(2.1) \quad |\Phi^p_{x,y/z}(t)| \leq \|y, z\| \quad \text{for all } t \in \mathbb{R}$$

(ii) We have the inequality

$$(2.2) \quad \frac{\|x + 2uy, z\| - \|x + uy, z\|}{u} \leq \Phi^i_{x,y/z}(u) \leq \Phi^s_{x,y/z}$$

and

$$(2.3) \quad \frac{\|x + 2ty, z\| - \|x + ty, z\|}{t} \leq \Phi^s_{x,y/z}(t) \leq \Phi^i_{x,y/z}(t)$$

(iii) The mapping $\Phi^p_{x,y/z}$ are strictly increasing on $\mathbb{R}$.

(iv) We have the limits

$$(2.4) \quad \lim_{u \to -\infty} \Phi^p_{x,y/z}(u) = \|y, z\|, \quad \lim_{t \to +\infty} \Phi^p_{x,y/z}(t) = \|y, z\|$$

and

$$(2.5) \quad \lim_{t \to 0^+} \Phi^p_{x,y/z}(t) = \frac{(y, x/z)_s}{\|x, z\|}, \quad \lim_{u \to 0^-} \Phi^p_{x,y/z}(u) = \frac{(y, x/z)_i}{\|x, z\|}$$

(v) The mapping $\Phi^s$ is right continuous and $\Phi^i$ is left continuous at every point of $\mathbb{R}$.
Proof. (i) Follows by the Schwarz inequality.
(ii) Let $u < 0$. By the Schwarz inequality (iii) and by properties (iv) and (ii) of 2-norm derivatives $(.,/z)_i$, we have
\[
\| x + 2uy, z \| \geq (x + 2uy, x + uy/z)_s = (x + uy + uy, x + uy/z)_s
\]
\[
= \| x + uy, z \|^2 - u(-y, x + uy/z)_s
\]
\[
= \| x + uy, z \|^2 + u(y, x + uy/z)_i.
\]
From which we get
\[
\| x + 2uy, z \| - \| x + uy, z \| \geq u(y, x + uy/z)_i.
\]
This implies
\[
\frac{\| x + 2uy, z \| - \| x + uy, z \|}{u} \leq \frac{(y, x + uy/z)_i}{\| x + uy, z \|}
\]
and the (i) inequality in (2.2) is proved.

Further,
\[
\| x, z \| \geq (x, x + uy/z)_s
\]
\[
= (x + uy - uy, x + uy/z)_s
\]
\[
= \| x + uy, z \|^2 + (-uy, x + uy/z)_s.
\]
From which we get
\[
\frac{\| x + uy, z \|^2 - \| x, z \|^2}{u} \geq \frac{(y, x + uy/z)_s}{\| x + uy, z \|} = \Phi_{x,y/z}^s(u).
\]
The (iii) inequality in (2.2) is proved.
Inequality (2.3) is proved similarly.
(iii) Suppose that $p \in \{i, s\}$ and $t_2 > t_1$. Then by Schwarz inequality
\[
\| x + t_2y, z \| \geq (x + t_2y, x + t_1y/z)_p
\]
for all $x, y \in X$ and $z \in X/L(\{x, y\}, G)$. Using properties of 2-norm derivatives, we obtain
\[
(x + t_2 y, x + t_1 y/z)_p \geq (t_2 - t_1/y + x + t_1 y/z)_p
\]
and the above inequality yields
\[
\| x + t_1 y, z \|^2 + (t_2 - t_1)(y, x + t_1 y/z)_p
\]
and
\[
\| x + t_2 y, z \| \geq \| x + t_1 y, z \|^2 + (t_2 - t_1)(y, x + t_1 y/z)_p
\]
Hence
\[
\Phi^p_{x,y/z}(t_1) = \frac{(y, x + t_1 y/z)_p}{\| x + t_1 y, z \|} \leq \frac{\| x + t_2 y, z \| - \| x - t_1 y, z \|}{t_2 - t_1}
\]
put $t = t_2 - t_1 > 0$ then by (2.3)
\[
\frac{\| x + t_2 y, z \| - \| x - t_1 y, z \|}{t_2 - t_1} = \frac{\| x + t_2 y + ty, z \|}{t} - \| x + t_1 y, z \|
\]
\[
\Phi^p_{x,y/z}(t_1) = \frac{(y, x + t_1 y/z)_p}{\| x + t_1 y, z \|} \leq \Phi^p_{x+t_1 y,y/z}(t) = \frac{(y, x + (t_1 y + ty)/z)_p}{\| x + t_1 y + ty, z \|}
\]
\[
= \frac{(y, x + t_2 y/z)_p}{\| x + t_2 y, z \|} = \Phi^p_{x,y/z}(t_2)
\]
and the statement is proved.

(iv) We have
\[
\lim_{t \to +\infty} \frac{\| x + ty, z \| - \| x, z \|}{t} = \lim_{\alpha \to 0^+} \frac{\| x + \frac{y}{\alpha}, z \| - \| x, z \|}{\frac{1}{\alpha}}
\]
\[
\lim_{\alpha \to 0^+} \frac{\| \alpha x + y, z \| - \alpha \| x, z \|}{t} = \| y, z \|
\]
and
\[
\lim_{t \to +\infty} \frac{\| x + 2ty, z \| - \| x + ty, z \|}{t} = \lim_{\alpha \to +\infty} \frac{\| t \| \frac{x}{t} + 2y, z \| - \| \frac{x}{t} + y, z \|}{t}
\]
Applying the inequality (2.3) we get the second limit in (2.4) the first limit is obtained similarly.

Further
\[
\lim_{t \to 0^+} \frac{\| x + ty, z \| - \| x, z \|}{t} = \lim_{t \to 0^+} \frac{\| x + ty, z \|^2 - \| x, z \|^2}{2t} \times \lim_{t \to 0^+} \frac{2}{\| x + ty, z \| + \| x, z \|} = \frac{(y, x/z)_s}{\| x, z \|}
\]
and

\[
\lim_{t \to 0^+} \frac{\| x + 2ty, z \| - \| x + ty, z \|}{t} = \lim_{t \to 0^+} \frac{\| x + 2ty, z \| - \| x, z \| - (\| x + ty, z \| - \| x, z \|)}{t}
\]
\[
= 2 \lim_{t \to 0^+} \frac{\| x + 2ty, z \| - \| x, z \|}{2t} - \lim_{t \to 0^+} \frac{\| x + ty, z \| - \| x, z \|}{t} - \frac{2(y, x/z)_s}{\| x, z \|} - \frac{(y, x/z)_s}{\| x, z \|} - \frac{(y, x/z)_s}{\| x, z \|}.
\]

Inequality (2.3) applied to these limit yields the first in (2.3); the second limit is obtained similarly.

(v) Let \( t_0 \in R \)

\[
\lim_{\alpha \to t_0^+} \Phi_{x,y/z}^p (\alpha) = \lim_{t \to 0^+} \Phi_{x,y/z}^p (t_0 + t) = \lim_{t \to 0^+} \frac{(y, x + t_0y + ty/z)_p}{\| x + t_0y + ty, z \|}
\]
\[
\lim_{t \to 0^+} \Phi_{x,y/z}^p (t) = \frac{(y, x + t_0y/z)_s}{\| x + t_0y, z \|} = \Phi_{x,y/z}^s (t_0)
\]
in the statement above the right continuity is proved. The statement about the left continuity is proved similarly.
3 New Characterizations of Birkhoff Orthogonality and Smoothness

The mapping $\Phi^p_{x,y/z}$ can be used to give a characterization of Birkhoff Orthogonality.

**Theorem 3.1.** Let $(X, \| \cdot \|)$ be a real normed linear space, and let $x,y$ be a two elements of $X$ and $\in X/L(\{x,y\},G)$. The following statement are equivalent
(i) $x \perp y(B)$
(ii) If $p,q \in \{i,s\}$ and $u < 0 < t$ then the following inequality holds:

\[(3.1) \quad \Phi^p_{x,y/z}(u) \leq 0 \leq \Phi^q_{x,y/z}(t)\]

**Proof.** We know that Birkhoff Orthogonality $x \perp y(B)$ is equivalent to the inequality

\[(3.2) \quad (y, x/z)_i \leq 0 \leq (y, x/z)_s\]

According to the Theorem 2.1, we have that

\[(3.3) \quad \Phi^p_{x,y/z}(u) \leq \frac{\|x + uy, z\| - \|x, z\|}{u}, \quad u < 0\]

\[(3.4) \quad \Phi^p_{x,y/z}(t) \geq \frac{\|x + ty, z\| - \|x, z\|}{t}, \quad t > 0\]

whenever $p \in \{s, i\}$.

(i) $\Rightarrow$ (ii) if $x \perp y(B)$, then $\|x + \alpha y, z\| \geq \|x, z\|$ for all $\alpha \in R$. Hence

\[\frac{\|x + uy, z\| - \|x, z\|}{u} \leq 0 \leq \frac{\|x + ty, z\| - \|x, z\|}{t}\]

for $u < 0 < t$. Using inequality (3.3) and (3.4) we get (3.1).

(ii) $\Rightarrow$ (i) According to the Theorem 2.1, we have that

\[\lim_{t \to 0^+} \Phi^p_{x,y/z}(t) = \frac{(y, x/z)_s}{\|x, z\|}, \quad \lim_{u \to 0^-} \Phi^p_{x,y/z}(u) = \frac{(y, x/z)_i}{\|x, z\|}.\]
If (3.1) holds then \((y, x/z)_s \geq 0 \geq (y, x/z)_i\) using (3.2) we deduce that \(x \perp_z y(B)\).

**Theorem 3.2.** Let \((X, \| \cdot \|)\) be a real 2-normed linear space and let \(x \in X \setminus \{0\}\). The following statements are equivalent

(i) \(X\) is smooth at \(x_0\),

(ii) The mapping \(\Phi^p_{x,y/z}\) is continuous at 0 for all \(y \in X\) and some \(p \in \{s, i\}\).

**Proof.** The space \(X\) is smooth at \(x_0\) if and only if the function \(x \to \| x, z \|\) is Gateaux differentiable at \(x_0\), this is equivalent to \((y, x_0/z)_i = (y, x_0/z)_s\) for all \(y \in X\) and \(z \in X/L(\{x, y\}, G)\). The equivalence of (i) and (ii) then follows in view of (2.5).

4 New Characterizations of Elements of Best Approximations in 2-norm Spaces

**Definition 4.1.** Let \(X\) be a 2-normed linear space, \(G\) a set in \(X\), and \(x \in X\). An element \(g_0 \in G\) is called an element of best approximation at \(x\), if

\[
\| x - g_0, z \| = \inf_{g \in G} \| x - g, z \|
\]

where \(z \in X/L(\{x, y\}, G)\).

We denote by \(P_{G,z}(X)\) the set at all such elements \(g_0\), that is

\[
P_{G,z}(x) = \{g_0 \in G \| x - g_0, z \| = \inf_{g \in G} \| x - g, z \|\}.
\]

It is of interest to consider the problem of finding necessary and sufficient conditions such that \(g_0 \in P_{g,z}(x)\).

**Lemma 4.1.** Let \((X, \| \cdot \|)\) be a 2-normed linear space, \(G\) a linear subspace of \(X\), \(x \in X \setminus \tilde{G}\) and \(g_0 \in G\). Then \(g_0 \in P_{G,z}(x)\) if and only if \(x - g_0 \perp_z G(B)\). The following preposition is true.
**Proposition 4.1.** Let \((X, \| \cdot \|)\) be a 2-normed linear space, \(G\) a linear subspace of \(X\), \(x \in X \setminus G\) and \(g_0 \in G\). The following statements are equivalent:

(i) \(g_0 \in P_{G,z}(x)\)

(ii) We have the equality

\[
\sup_{g \in G} (g + x - g_0, x - g_0/z) = \| x - g_0, z \|^2
\]

**Proof.** By Lemma 4.1, \(g_0 \in P_{G,z}(x)\) is equivalent to

\[x - g_0 \perp_z G(B)\]

and the property (viii) of the introduction to

\[
(g, x - g_0/z)_i \leq 0 \leq (g, x - g_0/z)_s \quad \text{for all } g \in G
\]

But

\[
(g, x - g_0/z)_i = (x - g_0 + g - x + g_0, x - g_0/z)_i
\]

\[
= \| x - g_0, z \|^2 + (g + x - g_0, x - g_0/z)_i
\]

and

\[
(g, x - g_0/z)_s = (x - g_0 + g - x + g_0, x - g_0/z)_s
\]

\[
= \| x - g_0, z \|^2 - (g + x - g_0, x - g_0/z)_s
\]

\[
= \| x - g_0 \|^2 - (g + x - g_0, x - g_0/z)_i
\]

Then (4.4) is equivalent to

\[
(g + x - g_0, x - g_0/z)_i \leq \| x - g_0, z \|^2 \quad \text{for all } g \in G
\]

\[
(-g + x - g_0, x - g_0/z)_i \leq \| x - g_0, z \|^2 \quad \text{for all } g \in G
\]

\(g \in G\) if and only if \(-g \in G\), we deduce that (4.4) is equivalent to (4.3) and the proposition is proved.
Lemma 4.2. Let \((X, \| \cdot \|)\) be a real 2-normed space and \(x, y\) two elements of \(X\) and \(z \in X/L(\{x, y\}; G)\). The following statements are equivalent:

(i) \(x \perp z\) y(B)

(ii) \((y, x + uy/z)_p \leq 0 \leq (y, x + ty/z)_q\) whenever \(u < 0 < t\) and \(p, q \in \{i, s\}\).

Using this Lemma, we obtain the following new characterization of best approximants in terms of the 2-norm derivatives.

Theorem 4.1. Let \(X, G, x\) and \(g\) be as in Proposition 4.1. The following statements are equivalent.

(i) \(g_0 \in P_{G, z}(x)\)

(ii) We have the inequality

\[(g, x - g_0 + ug/z)_p \leq \| x - g_0 + w, z \|^2 \quad \text{if} \quad w \in G, p \in \{i, s\}\]

Proof By Lemma 4.2, \(g_0 \in P_{G, z}(x)\) is equivalent to

\[(g, x - g_0 + ug/z)_p \leq 0 \leq (g, x - g_0 + tg/z)_q \quad \text{if} \quad u < 0 < t, q \in \{i, s\}\]

But

\[(g, x - g_0 + tg/z)_q \leq 0, \quad t > 0\]

is equivalent to

\[(tg, x - g_0 + tg/z)_q \geq 0, \quad t > 0\]

As

\[(tg, x - g_0 + tg/z)_q = (x - g_0 + tg - x + g_0, x - g_0 + tg/z)_q\]

\[= \| x - g_0 + tg, z \|^2 - (x - g_0, x - g_0 + tg/z)_r\]

with \(r \in \{i, s\}, r \neq q\) (4.9) is equivalent to

\[(x - g_0, x - g_0 + tg/z)_q \leq \| x - g_0 + tg, z \|^2\]
for all \( g \in G, t > 0, g \in \{i, s\} \).

The relation

\[
(4.11) \quad (g, x - g_0 + ug/z)_p \leq 0, u < 0, p \in \{i, s\}
\]

is equivalent to

\[
-u(g, x - g_0 + ug/z)_p \leq 0, p \in \{i, s\}.
\]

But

\[
-u(g, x - g_0 + ug/z)_p = (-ug, x - g_0 + ug/z)_p = -(ug, x - g_0 + ug/z),
\]

with \( r \in \{i, s\}, r \neq p \); hence (4.11) is equivalent to

\[
(ug, x - g_0 + ug/z)_p \geq 0, \quad p \in \{i, s\}, u < 0.
\]

On the other hand

\[
(ug, x - g_0 + ug/z)_p = (x - g_0 + ug - x + g_0, x - g_0 + ug/z)_p
\]

\[
= \|x - g_0 + ug, z\|^2 - (x - g_0, x - g_0 + ug/z),
\]

and (4.11) is equivalent to

\[
(4.12) \quad (x - g_0, x - g_0 + ug/z)_p \leq \|x - g_0 + ug, z\|^2
\]

for all \( g \in G, u < 0, p \in \{i, s\} \). Combining (4.10) and (4.12) and observing that (4.10) holds (with equality) also for \( t = 0 \), we conclude that

\[
(x - g_0, x - g_0 + tg/z)_p \leq \|x - g_0 + tg, z\|^2
\]

for all \( g \in G \) and all \( t \in R \).

As \( g \in G \) if and only if \( t, g \in G \) for \( t \neq 0 \), we deduce the desired equivalence, and the theorem is proved.
References


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