Kalecki’s model of business cycle. Data dependence

I. M. Olaru, C. Pumnea, A. Bacociu, A. D. Nicoară

Abstract

In this paper we study date dependence for a delay equation which models a business cycle. The study is made using weakly Picard operators.

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1 Introduction

Let $(X, d)$ be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset \},$$

the set of valid parts of $X$;

$$F_A := \{x \in X \mid A(x) = x \},$$

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the fixed point set of $A$;

$$ I(A) := \{ Y \in P(X) \mid A(Y) \subseteq Y \}, $$

the family of the nonempty invariant subset of $A$.

$$ A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in \mathbb{N}. $$

**Definition 1.1.** ([2],[3]) An operator $A$ is weakly Picard operator (WPO) if the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges, for all $x \in X$ and the limit (which depend on $x$) is a fixed point of $A$.

**Definition 1.2.** ([2],[3]) If the operator $A$ is weakly Picard operator and $F_A = \{ x^n \}$ then by definition $A$ is Picard operator.

**Definition 1.3.** ([2],[3]) If $A$ is weakly Picard operator, then we consider the operator

$$ A^\infty : X \longrightarrow X, A^\infty(x) = \lim_{n \to \infty} A^n(x). $$

**Observation 1.1.** $A^\infty(x) = F_A$.

**Definition 1.4.** ([2],[3]) Let be $A$ an WPO and $c > 0$. The operator $A$ is $c$-weakly Picard operator if

$$ d(x, A^\infty(x)) \leq c \cdot d(x, A(X)) $$

for all $x \in X$.

**Observation 1.2.** ([4]) If $(X,d)$ is a metric space and $A : X \longrightarrow X$ an operator is a $a$-contraction, then $A$ is $c$-weakly Picard operator with $c = \frac{1}{1-a}$.

The next result it is a characteristic of weakly Picard operator, respectively $c$-weakly Picard.

**Theorem 1.1.** ([2],[3]) Let $(X,d)$ be a metric space and $A : X \longrightarrow X$ an operator. The operator $A$ is weakly Picard operator (c-weakly Picard operator) if and only if there exists a partition of $X$,

$$ X = \bigcup_{\lambda \in \Lambda} X_\lambda $$

such that:
(a) $X_\lambda \in I(A)$;

(b) $A \mid X_\lambda : X_\lambda \longrightarrow X_\lambda$ is a Picard (c-Picard) operator, for all $\lambda \in A$.

2 Main result

We consider the equation

\begin{equation}
I'(t) = \frac{m}{t} I(t) - (\frac{m}{\tau} + n) I(t - \tau) + nu, \quad t \in [0, T]
\end{equation}

where $\tau > 0, m = \frac{p}{1 - \alpha}, p > 0, n > 0, \alpha \in (0, 1), u > 0$, with the initial condition

\begin{equation}
I(t) = \varphi(t), \quad t \in [-\tau, 0]
\end{equation}

with $\varphi : [-\tau, 0] \longrightarrow \mathbb{R}$.

The Cauchy problem (1) + (2) is equivalent with the next integral equation

\begin{equation}
I(t) = \begin{cases}
\varphi(0) + \int_0^t \left[ \frac{m}{\tau} I(s) - (\frac{m}{\tau} + \eta) I(s - \tau) + nu \right] ds, & t \in [0, T] \\
\varphi(t), & t \in [-\tau, 0]
\end{cases}
\end{equation}

We search the solutions for the integral equation (3) in the continuous functions space $(C[-\tau, T], \cdot, \cdot_r)$ endowed with Bielecki norm,

\[ |x|_r = \sup_{t \in [-\tau, T]} |x(t)| e^{-rt}. \]

Next, we consider the operator $A : C[-\tau, T] \longrightarrow C[-\tau, T]$, defined by

\begin{equation}
A(I)(t) = \begin{cases}
\varphi(0) + \int_0^t \left[ \frac{m}{\tau} I(s) - (\frac{m}{\tau} + \eta) I(s - \tau) + nu \right] ds, & t \in [0, T] \\
\varphi(t), & t \in [-\tau, 0]
\end{cases}
\end{equation}
Then for any $I_1, I_2 \in C[-\tau, T]$ we have

$$|A(I_1)(t) - A(I_2)(t)| \leq \int_0^t \left[ \frac{m}{\tau} |I_1(s) - I_2(s)| + \left( \frac{m}{\tau} + n \right) |I_1(s-\tau) - I_2(s-\tau)| \right] ds \leq$$

$$|I_1 - I_2|r \int_0^t \frac{m}{\tau} e^{rs} + \left( \frac{m}{\tau} + n \right) e^{r(s-\tau)} ds \leq$$

$$|I_1 - I_2|r e^{rt} \left( \frac{2m}{r} + n \right) e^{r(t-\tau)}.$$

It follows that

$$|A(I_1) - A(I_2)|_r \leq \frac{2m}{r} + n \cdot |I_1 - I_2|_r.$$

Using the Banach principle of fixed point it results that equation (3) has, in $C[-\tau, T]$, a unique solution $I^*(\cdot, \varphi)$.

In the following lines, using the characterization theorem of the weakly Picard operator we show that equation (1) has a infinity of solutions. Indeed, for $\varphi \in C[-\tau, 0]$, we consider

$$X_\varphi = \{ x \in C[-\tau, T] | x|_{[-\tau, 0]} = \varphi \}.$$

We choose $r > 0$ thus

$$\frac{2m}{\tau} + n < 1$$

it results

$$|A(I_1) - A(I_2)|_r \leq \left( \frac{2m}{r} + n \right) |I_1 + I_2|.$$

According to the above, the operator $A|_{X_\varphi} : X_\varphi \longrightarrow X_\varphi$ is a Picard operator. Then $A$ is a weakly Picard operator and as consequence the equation (1) has a infinity of solutions.

Next we assume that it exists $\eta > 0$ thus

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta.$$
Let $I^*(\cdot, \varphi_1), I^*(\cdot, \varphi_2)$ be the solution of the equation (3) with data $\varphi_1, \varphi_2$.

Then

$$|I^*(t, \varphi_1) - I^*(t, \varphi_2)| \leq$$

$$\leq \eta \int_0^t \left( \frac{m}{\tau} |I^*(s, \varphi_1) - I^*(s, \varphi_2)| + \left( \frac{m}{\tau} + n \right) |I^*(s - \tau, \varphi_1) - I^*(t, \varphi_2)| \right) ds.$$  

According with Theorem 14.6 (see [1] pp 145) we obtain

$$|I_1(t) - I_2(t)| \leq \eta k(m, n, \tau) e^{\int_0^t (\frac{2m}{\tau} + n) ds} \leq \eta k(m, n, \tau) e^{(\frac{2m}{\tau} + n)T}$$

So, from above we obtain the following result

**Theorem 2.1.** We consider the equation (1). Then:

(a) the equation (1) has, in $C([-\tau, T]), \cdot |_{r}$, an infinity of solutions;

(b) the problem (1)+(2) has, in $C([-\tau, T]), \cdot |_{r}, a unique solution $ I^*(\cdot, \varphi)$;

(c) if there exists $\eta > 0$ such that

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta,$$

for all $t \in [-\tau, T]$, then there exists $k(m,n,\tau) > 0$ such that

$$|I^*(t, \varphi_1) - I^*(t, \varphi_2)| \leq \eta k(m, n, \tau) e^{(\frac{2m}{\tau} + n)T}.$$  

**References**


Departament of Mathematics,
Faculty of Sciences,
University ”Lucian Blaga” of Sibiu,
Dr. Ion Ratiu 5-7, Sibiu, 550012, Romania