A sufficient condition for univalence
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Abstract
In this paper we obtain sufficient conditions for univalence, which generalize some well known univalence criteria for analytic functions in the unit disk.

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1 Introduction

We denote by $U_r = \{ z \in \mathbb{C} : |z| < r \}$ the disk of $z$-plane, where $r \in (0, 1]$, $U_1 = U$ and $I = [0, \infty)$. Let $A$ be the class of functions $f$ analytic in $U$ such that $f(0) = 0$, $f'(0) = 1$.

Theorem 1.1. (see [2]) Let $f \in A$. If for all $z \in U$

\begin{equation}
|\{f; z\}| \leq \frac{2}{(1 - |z|^2)^2}
\end{equation}

where

\begin{equation}
\{f; z\} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2
\end{equation}

then the function $f$ is univalent in $U$.  

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Theorem 1.2. (see [1]) Let $f \in A$. If for all $z \in U$

\begin{equation}
(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,
\end{equation}

then the function $f$ is univalent in $U$.

Theorem 1.3. (see [3]) Let $f \in A$. If for all $z \in U$

\begin{equation}
\left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < 1
\end{equation}

then the function $f$ is univalent in $U$.

2 Preliminaries

Our considerations are based on the theory of Löwner chains; we first recall the basic result of this theory, from Pommerenke.

Theorem 2.1. (see [4]) Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots$, $a_1(t) \neq 0$ be analytic in $U_r$, for all $t \in I$, locally absolutely continuous in $I$ and locally uniformly with respect to $U_r$. For almost all $t \in I$, suppose that

\[ z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t}, \quad \text{for all } z \in U_r, \]

where $p(z, t)$ is analytic in $U$ and satisfies the condition $\text{Re } p(z, t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z, t)/a_1(t)\}$ forms a normal family in $U_r$, then for each $t \in I$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk $U$.

3 Main results

Theorem 3.1. Let $\beta$ be a real number, $\beta > 1/2$ and $f \in A$. If there exist the analytic functions $g$ and $h$ in $U$, $g(z) = 1+b_1z+\ldots$, $h(z) = c_0+c_1z+\ldots$, such that the inequalities

\begin{equation}
\left| \frac{f'(z)}{g(z)} - \beta \right| < \beta
\end{equation}
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and

\begin{align}
(6) \quad \left| \left( \frac{f'(z)}{g(z)} - \beta \right) |z|^{2\beta} + (1 - |z|^{2\beta}) \left( \frac{2zf'(z)h(z)}{g(z)} + \frac{zg'(z)}{g(z)} + 1 - \beta \right) \\
+ \frac{(1 - |z|^{2\beta})^2}{|z|^{2\beta}} \left( \frac{z^2f'(z)h^2(z)}{g(z)} + \frac{z^2g'(z)h(z)}{g(z)} - z^2h'(z) \right) \right| \leq \beta
\end{align}

are true for all \( z \in U \), then the function \( f \) is univalent in \( U \).

**Proof.** The functions \( f, g, h \) being analytic in \( U \), it is easy to see that there is a real number \( r_1 \in (0, 1] \) such that the function

\begin{equation}
L(z, t) = f(e^{-t}z) + \frac{(e^{2\beta t} - 1) \cdot e^{-t}z \cdot g(e^{-t}z)}{1 + (e^{2\beta t} - 1) \cdot e^{-t}z \cdot h(e^{-t}z)}
\end{equation}

is analytic in \( U_{r_1} \), for all \( t \in I \). If \( L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots \) is the power series expansion of \( L(z, t) \) in the neighborhood \( U_{r_1} \), it can be checked that we have \( a_1(t) = e^{(2\beta - 1)t} \) and therefore \( a_1(t) \neq 0 \) for all \( t \in I \). From \( \beta > 1/2 \), it follows that \( \lim_{t \to \infty} |a_1(t)| = \infty \).

Since \( L(z, t)/a_1(t) \) is the summation between \( z \) and an analytic function, we conclude that \( \{L(z, t)/a_1(t)\}_{t \in I} \) is a normal family in \( U_{r_2} \), \( 0 < r_2 < r_1 \). By elementary computations, it can be shown that \( \frac{\partial L(z, t)}{\partial t} \) can be expressed as the summation between \((2\beta - 1)e^{(2\beta - 1)t}z \) and an analytic function in \( U_r \), \( 0 < r < r_2 \), and hence we obtain the absolute continuity requirement of Theorem 2.1. Let \( p(z, t) \) be the analytic function defined in \( U_r \) by

\begin{equation}
p(z, t) = z \frac{\partial L(z, t)}{\partial z} \left/ \frac{\partial L(z, t)}{\partial t} \right.
\end{equation}

In order to prove that the function \( p(z, t) \) has an analytic extension, with positive real part in \( U \), for all \( t \in I \), it is sufficient to show that the function \( w(z, t) \) defined in \( U_r \) by

\begin{equation}
w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}
\end{equation}
can be continued analytically in $U$ and that $|w(z, t)| < 1$ for all $z \in U$ and $t \in I$.

By simple calculations, we obtain

$$w(z, t) = \frac{1}{\beta} \left( \frac{f'(e^{-t}z)}{g(e^{-t}z)} - \beta \right) e^{-2\beta t} +$$

$$\frac{1 - e^{-2\beta t}}{\beta} \left( \frac{2e^{-t}zf'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)} + 1 - \beta \right) +$$

$$\frac{(1 - e^{-2\beta t})^2 e^{-2\beta t}}{\beta e^{-2\beta t}} \left( \frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z) \right).$$

From (5) and (6) we deduce that the function $w(z, t)$ is analytic in the unit disk $U$. From (5) and since $\beta > 1/2$ we have

$$|w(z, 0)| = \left| \frac{f'(z)}{g(z)} - \beta \right| < 1 \quad (9)$$

$$|w(0, t)| = \left| \frac{1 - \beta}{\beta} \right| < 1. \quad (10)$$

Let $t$ be a fixed number, $t > 0$ and observing that $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \overline{U} = \{ z \in C : |z| \leq 1 \}$ we conclude that the function $w(z, t)$ is analytic in $\overline{U}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in R$ such that

$$|w(z, t)| < \max_{|\xi|=1} |w(\xi, t)| = |w(e^{i\theta}, t)|, \quad (11)$$

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (8) we get

$$|w(e^{i\theta}, t)| = \frac{1}{\beta} \left| \left( \frac{f'(u)}{g(u)} - \beta \right) u^{2\beta} + (1 - |u|^{2\beta}) \right.$$}

$$\left( \frac{2uf'(u)h(u)}{g(u)} + \frac{ug'(u)}{g(u)} + 1 - \beta \right)$$

$$+ \frac{(1 - |u|^{2\beta})^2 u^2}{|u|^{2\beta}} \left( \frac{f'(u)h^2(u)}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u) \right).$$
The inequality (6) implies $|w(e^{i\theta}, t)| \leq 1$ and by using (9), (10) and (11) it follows that $|w(z, t)| < 1$ for all $z \in U$ and $t \geq 0$. From Theorem 2.1 we obtain that the function $L(z, t)$ has an analytic and univalent extension to the whole unit disk $U$, for all $t \geq 0$. For $t = 0$ we have $L(z, 0) = f(z)$, $z \in U$ and therefore the function $f$ is univalent in $U$.

Suitable choices of the functions $g$ and $h$ in Theorem 3.1 gives us various univalence criteria, between them being the very known Nehari’s criterion, Becker’s criterion and also Ozaki-Nunokawa’s criterion.

**Corollary 1.** Let $\beta$ be a real number, $\beta > 1/2$ and $f \in A$. If for all $z \in U$

$$
|1 - |z|^{2\beta}|^2 - z^2 \{f; z\}^2 - 2 + \beta| \leq \beta
$$

(12)

where $\{f; z\}$ is defined by (2), then the function $f$ is univalent in $U$.

**Proof.** It results from Theorem 3.1 with $g = f'$ and $h = \frac{1}{2} \frac{f''}{f'}$.

**Remark 1.** If we consider $\beta = 1$ in Corollary 1, the inequality (12) becomes (1) and then we obtain the univalence criterion due to Nehari [2].

**Corollary 2.** Let $\beta$ be a real number, $\beta > 1/2$ and $f \in A$. If for all $z \in U$

$$
|1 - |z|^{2\beta}|^2 - \frac{z f''(z)}{f'(z)} + 1 - \beta| \leq \beta
$$

(13)

then the function $f$ is univalent in $U$.

**Proof.** It results from Theorem 3.1 with $g = f'$ and $h = 0$.

**Remark 2.** If we consider $\beta = 1$ in Corollary 2, the inequality (13) becomes (3) and then we obtain the univalence criterion due to Becker [1].

**Corollary 3.** Let $\beta$ be a real number, $\beta > 1/2$ and $f \in A$. If for all $z \in U$

$$
\left| \left( \frac{z^2 f'(z)}{f^2(z)} - 1 \right) - (\beta - 1) \right| < \beta
$$

(14)
(15) \[ \left| \left( \frac{z^2 f'(z)}{f^2(z)} - 1 \right) - (\beta - 1)|z|^{2\beta} \right| < \beta |z|^{2\beta} \]

then the function \( f \) is univalent in \( U \).

**Proof.** It results from Theorem 3.1 with \( g(z) = \left( \frac{f(z)}{z} \right)^2 \) and \( h(z) = \frac{1}{z} - \frac{f(z)}{z^2} \).

**Remark 3.** If we consider \( \beta = 1 \) in Corollary 3, the inequalities (14) and (15) become (4) and then we obtain the univalence criterion due to Ozaki and Nunokawa [3].

**References**


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