On some subclasses of starlike and convex functions

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Abstract

Throughout this paper, in the second section, we prove that if $f \in A$, $\alpha \geq 0$ and $F(z) = zf'(z)\left(\alpha + \frac{zf'(z)}{f(z)}\right)$ is starlike then $f$ is a starlike function and, in the third section, we prove that if $\alpha \in [0,1)$, $f \in A$ and $F(z) = zf'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)$ is starlike of order $\alpha$ then $f$ is a convex function of order $\alpha$.

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1 Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and $H(U) = \{f : U \rightarrow \mathbb{C} : f$ is holomorphic in $U\}$.

We will also use the following notations:

$H[a,n] = \{f \in H(U) : f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots\}$ for $a \in \mathbb{C}$, $n \in \mathbb{N}^*$,

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\[ A_n = \{ f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots \}, \ n \in \mathbb{N}^*, \] and for \( n = 1 \) we denote \( A_1 \) by \( A \) and this set is called the class of analytic functions normalized in the origin.

Let \( S \) be the class of holomorphic and univalent functions on the unit disc which are normalized with the conditions \( f(0) = 0, \ f'(0) = 1, \) so

\[ S = \{ f \in A : f \text{ is univalent in } U \}. \]

**Definition 1.1.** ([3]) Let \( f : U \to \mathbb{C} \) be a holomorphic function with \( f(0) = 0. \) We say that \( f \) is starlike in \( U \) with respect to zero (or, in brief, starlike) if the function \( f \) is univalent in \( U \) and \( f(U) \) is a starlike domain with respect to zero, meaning that for each \( z \in U \) the segment between the origin and \( f(z) \) lies in \( f(U). \)

**Theorem 1.1.** ([3]) (the theorem of analytical characterization of starlikeness) Let \( f \in H(U) \) be a function with \( f(0) = 0. \) Then \( f \) is starlike if and only if \( f'(0) \neq 0 \) and

\[ \text{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U. \]

Let \( S^* \) be the class of normalized starlike functions on the unit disc \( U, \) so

\[ S^* = \left\{ f \in A : \text{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U \right\}. \]

**Definition 1.2.** ([3]) Let \( f : U \to \mathbb{C} \) be a holomorphic function. We say that \( f \) is convex on \( U \) (or, in brief, convex) if \( f \) is univalent in \( U \) and \( f(U) \) is a convex domain.

**Theorem 1.2.** ([3]) (the theorem of analytical characterization of convexity) Let \( f \in H(U). \) Then \( f \) is convex if and only if \( f'(0) \neq 0 \) and

\[ \text{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U. \]
Let $K$ be the class of normalized convex functions on the unit disc $U$ and $K(\alpha)$ be the class of normalized convex functions of order $\alpha$, i.e.

$$K(\alpha) = \left\{ f \in A : \text{Re} \frac{zf''(z)}{f'(z)} + 1 > \alpha, \ z \in U \right\}.$$ 

**Lemma 1.1.** ([2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be a function that satisfies the condition

$$\text{Re} \psi(\rho i, \sigma, \mu + iv; z) \leq 0,$$

when $\rho, \sigma, \mu, \nu \in \mathbb{R}, \sigma \leq -\frac{n}{2}(1 + \rho^2), \sigma + \mu \leq 0$, for $z \in U, n \geq 1$.

If $p \in H[1,n]$ and

$$\text{Re} \psi(p(z),zp'(z),z^2p''(z); z) > 0, \quad z \in U$$

then

$$\text{Re} p(z) > 0, \quad z \in U.$$ 

**Definition 1.3** (1). Let $\alpha, \beta \in \mathbb{R}, n \in \mathbb{N}^*, f \in A_n$ with

$$\frac{f(z)f'(z)}{z} \neq 0, \ 1 - \alpha + \alpha \frac{zf'(z)}{f(z)} \neq 0, \ z \in U.$$

We say that the function $f$ is in the class $M_n^{\alpha,\beta}$ if the function $F : U \to \mathbb{C}$, defined as

$$F(z) = f(z)\left[\frac{zf''(z)}{f(z)}\right]^\alpha(1-\beta) \cdot \left[1 - \alpha + \alpha \frac{zf'(z)}{f(z)}\right]^\beta$$

is a starlike function on the unit disc $U$.

**Remark 1.1.** ([1])

1. If $\beta = 0$ then $F(z) = f(z)\left[\frac{zf''(z)}{f(z)}\right]^\alpha, \ z \in U$ and $M_{\alpha,0}^1 = M_{\alpha}$ (the class of $\alpha$-convex functions).

2. If $\beta = 1$ then $F(z) = (1 - \alpha)f(z) + \alpha zf'(z), \ z \in U$ and $M_{\alpha,1}^1 = P_{\alpha}$ (the class of $\alpha$-starlike functions defined by N.N. Pascu).
3. If $\alpha = 0$ then $F(z) = f(z)$, $z \in U$ and $M_{0,\beta}^1 = S^*$ (the class of starlike functions).

4. If $\alpha = 1$ then $F(z) = zf'(z)$, $z \in U$ and $M_{1,\beta}^1 = K$ (the class of convex functions).

**Remark 1.2.** ([1]) For all real numbers $\alpha, \beta$ satisfying the condition $\alpha \beta (1 - \alpha) \geq 0$ we have

$$M_{\alpha,\beta}^n \subset S^*.$$

## 2 A subclass of starlike functions

**Definition 2.1.** Let $\alpha \geq 0$ and $f \in A$ such that

$$\frac{f(z)f'(z)}{z} \neq 0, \alpha + \frac{zf'(z)}{f(z)} \neq 0, z \in U.$$

We say that the function $f$ is in the class $N_\alpha$ if the function $F : U \to \mathbb{C}$ given by

$$F(z) = z f'(z) \left( \alpha + \frac{zf'(z)}{f(z)} \right)$$

is starlike in $U$.

**Theorem 2.1.** For each real number $\alpha \geq 0$ we have

$$N_\alpha \subset S^*.$$

**Proof.** Let $f \in N_\alpha$, $f \in A$ with $\frac{f(z)f'(z)}{z} \neq 0$ and $\alpha + \frac{zf'(z)}{f(z)} \neq 0$, $z \in U$.

We denote $\frac{zf'(z)}{f(z)} = p(z)$, $z \in U$. We have $p \in H[1,1]$ and $F(z) = zf'(z) \cdot (\alpha + p(z))$. (We make the remark that $F(0) = 0$ and $F'(0) = \alpha + 1 \neq 0$).

For $z \in U \setminus \{0\}$ we apply the logarithm to the equality $F(z) = zf'(z)(\alpha + p(z))$ and we obtain:

$$\log F(z) = \log z + \log f'(z) + \log(\alpha + p(z)).$$
If we derive the above equality (with respect to the independent variable \( z \)) and, afterwards, we multiply the result with \( z \), we will obtain:

\[
(1) \quad \frac{zF'(z)}{F(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{\alpha + p(z)}.
\]

But \( \frac{zf'(z)}{f(z)} = p(z) \) implies that \( zf'(z) = p(z)f(z) \) and deriving this equality we obtain

\[
f'(z) + zf''(z) = p'(z)f(z) + p(z)f'(z) \quad |: \quad f'(z) \neq 0,
\]

so

\[
1 + \frac{zf''(z)}{f'(z)} = p'(z) \cdot z \cdot \frac{1}{p(z)} + p(z).
\]

We will replace the last equality in (1) and we will have:

\[
\frac{zF'(z)}{F(z)} = \frac{zp'(z)}{p(z)} + p(z) + \frac{zp'(z)}{\alpha + p(z)}, \quad z \in U \setminus \{0\}.
\]

We make the remark that the above equality is also verified for \( z = 0 \).

We denote

\[
(2) \quad \psi(p(z),zp'(z);z) = p(z) + zp'(z)\left(\frac{1}{p(z)} + \frac{1}{\alpha + p(z)}\right)
\]

From Definition 2.1 we know that the function \( F \) is starlike, so

\[
(3) \quad \text{Re} \frac{zF'(z)}{F(z)} > 0, \quad z \in U.
\]

Using the notation (2) the condition (3) is equivalent with

\[
\text{Re} \psi(p(z),zp'(z);z) > 0, \quad z \in U.
\]

Making the calculus we have:

\[
\text{Re} \psi(is,t) = \text{Re} \left[ is + t\left(\frac{1}{is} + \frac{1}{\alpha + is}\right)\right] =
\]
\[ \text{Re} \left[ is + t \left( \frac{-is}{s^2} + \frac{\alpha - is}{\alpha^2 + s^2} \right) \right] = \frac{t \alpha}{\alpha^2 + s^2} \leq \frac{-\alpha(1 + s^2)}{2(\alpha^2 + s^2)} \leq 0, \]

for all \( t \leq -\frac{1}{2}(1 + s^2) \) and \( s \in \mathbb{R} \).

Consequently, we have obtained \( \text{Re} \psi(is, t) \leq 0 \) for all \( s \in \mathbb{R} \) and \( t \leq -\frac{1 + s^2}{2} \) and

\[ \text{Re} \psi(p(z), zp'(z); z) > 0, \quad z \in U, \quad p \in H[1, 1], \]

from where it results that

\[ \text{Re} p(z) > 0, \quad z \in U. \]

So, returning to the notation \( \frac{zf'(z)}{f(z)} = p(z) \) we obtain

\[ \text{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U, \]

and that means that \( f \in S^* \). So, \( N_\alpha \subset S^* \).

### 3 A subclass of convex functions of order \( \alpha \)

**Definition 3.1.** Let \( \alpha \in [0, 1) \) and \( f \in A \) with

\[ \frac{f(z)f'(z)}{z} \neq 0, \quad 1 + \frac{zf''(z)}{f'(z)} \neq 0, \quad z \in U. \]

We say that the function \( f \) is in the class \( N(\alpha) \) if the function \( F : U \to \mathbb{C} \) given by

\[ F(z) = zf'(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right), \]

is starlike of order \( \alpha \).

**Theorem 3.1.** For \( \alpha \in [0, 1) \) we have

\[ N(\alpha) \subset K(\alpha). \]
Proof. Let \( f \in N(\alpha) \). We denote \( 1 + \frac{zf''(z)}{f'(z)} = (1 - \alpha)p(z) + \alpha p(z) \). We have \( p \in H[1, 1] \) and \( F(z) = zf'(z)[(1 - \alpha)p(z) + \alpha] \). Using the logarithmic derivation and the multiplying with \( z \) we obtain:

\[
\frac{zF'(z)}{F(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{(1 - \alpha)p'(z) \cdot z}{(1 - \alpha)p(z) + \alpha} =
\]

\[
= (1 - \alpha)p(z) + \alpha + \frac{zp'(z)(1 - \alpha)}{(1 - \alpha)p(z) + \alpha}
\]

which is equivalent with

\[
(4) \quad \frac{zF'(z)}{F(z)} - \alpha = (1 - \alpha)p(z) + \frac{(1 - \alpha)zp'(z)}{(1 - \alpha)p(z) + \alpha}.
\]

We denote

\[
(5) \quad \psi(p(z),zp'(z);z) = (1 - \alpha)p(z) + \frac{zp'(z)(1 - \alpha)}{(1 - \alpha)p(z) + \alpha}, \ z \in U.
\]

We know that \( f \in N(\alpha) \), so \( F \) is starlike of order \( \alpha \), and hence

\[
(6) \quad \text{Re} \frac{zF'(z)}{F(z)} > \alpha, \ z \in U.
\]

Using (4) and the notation (5), the condition (6) is equivalent with

\[
\text{Re} \psi(p(z),zp'(z);z) > 0, \ z \in U.
\]

Making the calculus we have

\[
\text{Re} \psi(is,t) = \text{Re} \left[ (1 - \alpha)is + \frac{t(1 - \alpha)}{(1 - \alpha)is + \alpha} \right] =
\]

\[
= \frac{\alpha(1 - \alpha)t}{(1 - \alpha)^2s^2 + \alpha^2} \leq -\frac{\alpha(1 - \alpha)(1 + s^2)}{2[(1 - \alpha)^2s^2 + \alpha^2]} \leq 0
\]

for \( \alpha \in [0, 1) \), \( s \in \mathbb{R} \) and \( t \leq -\frac{1}{2}(1 + s^2) \).
Consequently, we have obtained $\Re \psi(is, t) \leq 0$ for all $s \in \mathbb{R}$ and $t \leq -\frac{1 + s^2}{2}$ and

$$\Re \psi(p(z), zp'(z); z) > 0, \ z \in U, \ p \in H[1, 1],$$

from where it results that

$$\Re p(z) > 0, \ z \in U.$$

Returning to the notation $1 + \frac{zf''(z)}{f'(z)} = (1 - \alpha)p(z) + \alpha$ and using the inequality $\Re p(z) > 0, \ z \in U$ we obtain $\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) = (1 - \alpha)\Re p(z) + \alpha > \alpha$ for $\alpha \in [0, 1)$, so $f \in K(\alpha)$.

Finally we have $N(\alpha) \subset K(\alpha)$.

References


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