A note on a general integral operator of the bounded boundary rotation$^1$

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Abstract

In this note, we consider the classes of bounded radius rotations, bounded radius rotation of order $\beta$, bounded boundary rotation. In these classes we study some properties of a general integral operator.

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1 Introduction

Let $\mathcal{P}_k^\lambda(\beta)$ denote the class of analytic functions $p(z)$ in defined in the unit disc $U = \{z : |z| < 1\}$ with the following properties:

(i). $p(0) = 1$

(ii). $\int_0^{2\pi} \left| \Re\left\{ e^{i\lambda} p(z) - \beta \cos \lambda \right\} \right| \frac{1}{1 - \beta} d\theta \leq k\pi \cos \lambda$

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where, \( k \geq 2 \), \( \lambda \) real, \(|\lambda| < \frac{\pi}{2}\), \( 0 \leq \beta < 1 \) and \( z = re^{i\theta} \) for \( 0 \leq r < 1 \).

Let \( \mathcal{V}_k^\lambda(\beta) \) [4] denote the class of functions \( f \) analytic in \( U \) with the normalized properties \( f(0) = f'(0) - 1 = 0 \) and

\[ 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\beta), \quad z \in U \]

where, \( k, \lambda \) and \( \beta \) are as above. For \( \beta = 0 \) we get the class \( \mathcal{V}_k^\lambda \) of functions with bounded boundary rotation studied by Moulis [3].

Any function \( f \in \mathcal{V}_k^\lambda(\beta) \) if and only if

\[ \Re \left\{ e^{i\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta \cos \lambda, \quad \text{for} \quad |z| < \frac{k - \sqrt{k^2 - 4}}{2}. \]

A function \( f \) defined in \( U \) with the normalization properties \( f(0) = 0 \) and \( f'(0) = 1 \) is said to be in the class \( \mathcal{U}_k^\lambda(\beta) \) if \( \frac{zf'}{f} \in \mathcal{P}_k^\lambda(\beta) \).

From the definition of the above classes it follows that \( f \in \mathcal{V}_k^\lambda(\beta) \) if and only if \( zf' \in \mathcal{U}_k^\lambda(\beta) \).

Now we consider the integral operator \( F_n(z) \) [2], defined by

\[ (1.1) \quad F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} \, dt \]

and we study its properties.

**Remark 1.1.** We observe that for \( n = 1 \) and \( \alpha_1 = 1 \), we obtain the integral operator of Alexander [1], \( F(z) = \int_0^z \frac{f(t)}{t} \, dt \).

## 2 Main results

**Theorem 2.1.** Let \( \alpha_i \) be real numbers with the properties \( 0 \leq \alpha_i < 1 \) for \( i \in \{1, 2, \ldots, n\} \) and \( \sum_{i=1}^n \alpha_i \leq n + 1 \). If \( f_i \in \mathcal{U}_k^\lambda \left( \frac{1}{\alpha_i} \right) \) then the integral operator defined in (1.1) belongs to \( \mathcal{V}_k^\lambda \).
Proof. Consider,

$$F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} dt.$$ 

We determine the derivatives of the first and second order for $F_n$.

$$F'_n(z) = \left( \frac{f_1(z)}{z} \right)^{\alpha_1} \cdots \left( \frac{f_n(z)}{z} \right)^{\alpha_n}$$

$$F''_n(z) = \sum_{i=1}^{n} \alpha_i \left( \frac{f_i(z)}{z} \right)^{\alpha_i-1} \frac{zf'_i(z) - f_i(z)}{z^2} \prod_{j=1, j\neq i}^{n} \left( \frac{f_j(z)}{z} \right)^{\alpha_j}$$

$$\frac{z F''_n(z)}{F'_n(z)} + 1 = \alpha_1 \frac{zf''_1(z)}{f'_1(z)} + \cdots + \alpha_n \frac{zf''_n(z)}{f'_n(z)} - \alpha_1 - \cdots - \alpha_n + 1$$

$$\Re \left\{ e^{i\lambda} \left( \frac{zf''_n(z)}{F'_n(z)} + 1 \right) \right\} = \alpha_1 \Re \left\{ e^{i\lambda} \frac{zf''_1(z)}{f'_1(z)} \right\} + \cdots + \alpha_n \Re \left\{ e^{i\lambda} \frac{zf''_n(z)}{f'_n(z)} \right\}$$

$$+ \Re \left\{ e^{i\lambda} (-\alpha_1 - \cdots - \alpha_n + 1) \right\}$$

$$= (n+1) \cos \lambda - \sum_{i=1}^{n} \alpha_i \cos \lambda > 0.$$ 

Hence $F_n \in \mathcal{V}_k^\lambda$.

Corollary 2.2. For parametric values $k = 2$, $\lambda = 0$, we get the following result [2].
Let $\alpha_i$, $i \in \{1, 2, \ldots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^{n} \alpha_i \leq n + 1$. We suppose that the functions $f_i$,
\( i \in \{1, 2, \ldots, n\} \) are starlike functions of order \( \frac{1}{\alpha_i}, \ i \in \{1, 2, \ldots, n\} \), that is \( f_i \in S^\ast \left( \frac{1}{\alpha_i} \right) \) for all \( i \in \{1, 2, \ldots, n\} \). Then the integral operator defined in (1.1) is convex.

**Theorem 2.3.** Let \( \alpha_i \) be real numbers with the properties \( \alpha_i > 0 \) for \( i \in \{1, 2, \ldots, n\} \) with \( \sum_{i=1}^{n} \alpha_i \leq 1 \) and \( f_i \in U^\lambda_k \left( \frac{1}{\alpha_i} \right) \). Then the integral operator defined in (1.1) belongs to \( V^\lambda_k(\alpha) \), where \( \alpha = 1 - \sum_{i=1}^{n} \alpha_i \).

**Proof.** Consider,
\[
\frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right)
= \sum_{i=1}^{n} \alpha_i \frac{zf'_i(z)}{f_i(z)} - \alpha_1 - \ldots - \alpha_n.
\]
\[
1 + \frac{zF''_n(z)}{F'_n(z)} = \alpha_1 \frac{zf'_1(z)}{f_1(z)} + \ldots + \alpha_n \frac{zf'_n(z)}{f_n(z)} - \alpha_1 - \ldots - \alpha_n + 1.
\]
\[
\Re \left\{ e^{i\lambda} \left( \frac{zF''_n(z)}{F'_n(z)} + 1 \right) \right\} = \alpha_1 \Re \left\{ e^{i\lambda} \frac{zf'_1(z)}{f_1(z)} \right\} + \ldots + \alpha_n \Re \left\{ e^{i\lambda} \frac{zf'_n(z)}{f_n(z)} \right\}
+ \Re \left\{ e^{i\lambda} (-\alpha_1 - \ldots - \alpha_n + 1) \right\}.
\]
But \( f_i \in U^\lambda_k \) for all \( i \in \{1, 2, \ldots, n\} \). Therefore
\[
\Re \left\{ e^{i\lambda} \frac{zf'_i(z)}{f_i(z)} \right\} > 0, \ \forall \ i \in \{1, 2, \ldots, n\}.
\]
This implies,
\[
\Re \left\{ e^{i\lambda} \left( \frac{zF''_n(z)}{F'_n(z)} + 1 \right) \right\} > 1 - \sum_{i=1}^{n} \alpha_i = \alpha.
\]
Hence \( F_n \in U^\lambda_k(\alpha) \).
**Corollary 2.4.** For parametric values $k = 2, \lambda = 0$, we get the following result [2].

Let $\alpha_i, i \in \{1, 2, ..., n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, 2, ..., n\}$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. We suppose that the functions $f_i$, with $i \in \{1, 2, ..., n\}$ are starlike. Then the integral operator defined in (1.1) is convex by order $1 - \sum_{i=1}^{n} \alpha_i$.

**References**


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