Several inequalities about the number of positive divisors of a natural number \( m \)

Nicușor Minculete\(^1\), Petrică Dicu\(^2\)

Abstract

In this paper we intend to establish several properties of the number of positive divisors of a natural number \( m \). Among these, we remark the inequality 
\[
\tau^2(mn) \geq \tau(m^2)\tau(n^2),
\]
for all \( m, n \in \mathbb{N}^* \).

2000 Mathematics Subject Classification: 11A25

Key words and phrases: prime number, arithmetic inequality, the number of natural divisors of \( m \).

1 Introduction

For a positive integer \( m \) number, we will note \( \tau(m) \) the number of positive divisors of \( m \). We remark that: \( \tau(1) = 1 \) and if \( p \) is a prime number, then
\[
\tau(p) = 2, \quad \tau(p^\alpha) = \alpha + 1.
\]

In papers [1]–[5], [7] we find the following properties of \( \tau(m) \):

For \( m = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}, m > 1 \) we have the relation:

\[
\tau(m) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_r + 1).
\]

\(^1\) Received 5 May, 2008

Accepted for publication (in revised form) 10 December, 2008
If \((m, n) = 1\), then

\[(2)\] \(\tau(mn) = \tau(m)\tau(n)\), for all \(m, n \in \mathbb{N}^*\).

For \(m \geq 2\), we have the relation:

\[(3)\] \(\tau(m) = \sum_{k=1}^{m} \left(\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{m-1}{k} \right\rfloor \right)\).

In [6], for \(m \geq 1\), we have

\[(4)\] \(\tau(m) \leq 2\sqrt{m}\).

In [8] it is shown that

\[(5)\] \(\tau(m)\tau(n) \geq \tau(mn)\), for all \(m, n \in \mathbb{N}^*\).

In [9] are establish the following inequalities:

\[(6)\] \(\tau(m) < m^{\frac{2}{3}}\), for any \(m > 12\),

\[(7)\] \(\ln \tau(m) < 1.066 \frac{\ln m}{\ln \ln m}\), for any \(m \geq 3\).

In this paper, we establish some new inequalities for the function \(\tau\).

### 2 Main results

We can remark several properties of these functions for two natural non-zero numbers, \(m\) and \(n\).

**Theorem 2.1.**

\[(8)\] a) \(\tau(mn) \leq \tau(m)n\), for all \(m, n \in \mathbb{N}^*\),

\[(9)\] b) \(n|m\), atunci \(\frac{\tau(m)}{m} \leq \frac{\tau(n)}{n}\), for all \(m, n \in \mathbb{N}^*\).
Proof. We will show that \( \tau(m) \leq m \), for all \( m \in \mathbb{N}^* \). From the inequality (4), \( \tau(m) \leq 2\sqrt{m} \), but \( m \geq 2\sqrt{m} \) for \( m \geq 4 \), therefore \( \tau(m) \leq m \), \( m \geq 4 \).

For \( m \in \{1, 2, 3\} \) it is easy to see that the inequality is true.

From the inequality (5), \( \tau(m) \tau(n) \geq \tau(mn) \), for all \( m, n \in \mathbb{N}^* \), but \( \tau(n) \leq n \), so \( \tau(mn) \leq \tau(m)n \), for all \( m, n \in \mathbb{N}^* \).

Because \( n|m \), we have \( m = nd \), and from the inequality (8) we obtain \( \tau(nd) \leq \tau(n)d \), which is equivalent with \( n\tau(m) \leq nd\tau(n) = m\tau(n) \).

Corollary 2.1. We have

(10) \[
\frac{\tau(mn)}{mn} \leq \frac{\tau(m) + \tau(n)}{m + n}, \text{ for all } m, n \in \mathbb{N}^*,
\]

(11) \[
b\tau(mn) \leq \frac{m^2\tau(n) + n^2\tau(m)}{m + n}, \text{ for all } m, n \in \mathbb{N}^*.
\]

Proof. We apply the inequality (8) and we deduce \( (m + n)\tau(mn) = m\tau(mn) + n\tau(mn) \leq mn\tau(m) + mn\tau(n) = mn(\tau(m) + \tau(n)) \), which means that the proof is complete.

Similarly, we prove the inequality \( (m + n)\tau(mn) = m\tau(mn) + n\tau(mn) \leq m^2\tau(n) + n^2\tau(m) \), consequently the inequality (11).

Theorem 2.2.

(12) \[
\tau((m, n))\tau([m, n]) = \tau(m)\tau(n), \text{ for all } m, n \in \mathbb{N}^*,
\]

where \( (m, n) \) is the greatest common divisor of \( m \) and \( n \) and \( [m, n] \) is the least common multiple of \( m \) and \( n \).

Proof. Let \( m \) and \( n \) be two natural non-zero numbers. We will factorize the numbers \( m \) and \( n \) in prime factors, thus:

\[
m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \cdot q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}, \quad n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k} \cdot r_1^{\delta_1} r_2^{\delta_2} \cdots r_t^{\delta_t}, \quad q_j \neq r_l, \text{ for all } j \in \{1, \ldots, s\} \text{ and for all } l \in \{1, \ldots, t\},
\]

therefore \( \tau(m) = \prod_{i=1}^{k} (\alpha_i + 1) \prod_{j=1}^{s} (\beta_j + 1) \), \( \tau(n) = \prod_{i=1}^{k} (\gamma_i + 1) \prod_{l=1}^{t} (\delta_l + 1) \), we obtain \( \tau((m, n)) = \prod_{i=1}^{k} (\min\{\alpha_i, \gamma_i\} + 1) \)
and \( \tau([m,n]) = \prod_{i=1}^{k} (\max\{\alpha_i, \gamma_i\} + 1) \prod_{j=1}^{s} (\beta_j + 1) \prod_{l=1}^{t} (\delta_l + 1) \), which means that \( \tau((m,n))\tau([m,n]) = \tau(m)\tau(n) \), for all \( m, n \in \mathbb{N}^* \).

**Theorem 2.3.**

(13) \( \tau^2(mn) \geq \tau(m^2)\tau(n^2) \), for all \( m, n \in \mathbb{N}^* \).

**Proof.** We consider \( m = \prod_{i=1}^{k} p_i^{\alpha_i} q_j^{\beta_j} \), \( n = \prod_{l=1}^{t} p_i^{\gamma_i} r_l^{\delta_l} \), which means that \( mn = \prod_{i=1}^{k} p_i^{\alpha_i+\gamma_i} q_j^{\beta_j} r_l^{\delta_l} \), hence \( \tau(m) = \prod_{i=1}^{k} (\alpha_i+1) \prod_{j=1}^{s} (\beta_j+1) \) and \( \tau(n) = \prod_{i=1}^{k} (\gamma_i+1) \prod_{l=1}^{t} (\delta_l+1) \), therefore \( \tau(mn) = \prod_{i=1}^{k} (\alpha_i+\gamma_i+1) \prod_{j=1}^{s} (\beta_j+1) \prod_{l=1}^{t} (\delta_l+1) \), so \( \tau(m)\tau(n) = \tau(mn) \cdot \prod_{i=1}^{k} (\alpha_i+1)(\gamma_i+1) = \tau(mn) \cdot \prod_{i=1}^{k} \left( 1 + \frac{\alpha_i\gamma_i}{\alpha_i+\gamma_i+1} \right) \geq \tau(mn) \). Because \( \tau(m^2) = \prod_{i=1}^{k} (2\alpha_i+1) \prod_{j=1}^{s} (2\beta_j+1) \) and \( \tau(n^2) = \prod_{i=1}^{k} (2\gamma_i + 1) \prod_{l=1}^{t} (2\delta_l + 1) \), we obtain the equality:

\[
\tau(m^2)\tau(n^2) = \prod_{i=1}^{k} (2\alpha_i+1) \prod_{j=1}^{s} (2\beta_j+1) \prod_{l=1}^{t} (2\gamma_i + 1) \prod_{l=1}^{t} (2\delta_l + 1),
\]

but \( \tau^2(mn) = \prod_{i=1}^{k} (\alpha_i+\gamma_i+1)^2 \prod_{j=1}^{s} (\beta_j+1)^2 \prod_{l=1}^{t} (\delta_l+1)^2 \). It is easy to see the equality \( \tau^2(mn) = \tau(m^2)\tau(n^2) \cdot \prod_{i=1}^{k} \left( 1 + \frac{(\alpha_i+\gamma_i)^2}{(2\alpha_i+1)(2\gamma_i+1)} \right) \cdot \prod_{j=1}^{s} \left( 1 + \frac{\beta_j^2}{2\beta_j+1} \right) \cdot \prod_{l=1}^{t} \left( 1 + \frac{\delta_l^2}{2\delta_l+1} \right) \).

Since \( 1 + \frac{(\alpha_i+\gamma_i)^2}{(2\alpha_i+1)(2\gamma_i+1)} \geq 1 \), for all \( i = \overline{1,k} \), \( 1 + \frac{\beta_j^2}{2\beta_j+1} \geq 1 \), for all \( j = \overline{1,s} \), \( 1 + \frac{\delta_l^2}{2\delta_l+1} \geq 1 \), for all \( l = \overline{1,t} \), we obtain \( \tau^2(mn) \geq \tau(m^2)\tau(n^2) \).

**Theorem 2.4.** Let \( m \) and \( n \) be two natural non-zero numbers, then \( \tau(mn) \leq n\sqrt{m} + m\sqrt{n} \).
Proof. We apply the inequality (4) for $m$ and $n$, we have $n\tau(m) \leq 2n\sqrt{m}$ and $m\tau(n) \leq 2m\sqrt{n}$. By adding the inequalities, we obtain

$$n\tau(m) + m\tau(n) \leq 2n\sqrt{m} + 2m\sqrt{n},$$

but using the inequality (8), we have $\tau(mn) \leq \tau(m)n$ and $\tau(mn) \leq \tau(n)m$, for all $m, n \in \mathbb{N}^*$, we deduce

$$2\tau(mn) \leq \tau(m)n + \tau(n)m,$$

so, from the inequalities (14) and (15), we obtain the inequality

$$\tau(mn) \leq n\sqrt{m} + m\sqrt{n}.$$

References


1University "Dimitrie Cantemir" of Brașov  
Department of REI Str. Bisericii Române, Nr. 107,  
Brașov, Romania.  
E-mail address: minculeten@yahoo.com

2University "Lucian Blaga" of Sibiu  
Department of Mathematics  
Str. Dr. I. Rațiu, Nr. 5–7,  
550012 - Sibiu, Romania.  
E-mail address: petrica.dicu@ulbsibiu.ro