Some theorems on the explicit evaluations of singular moduli

K. Sushan Bairy

Abstract

At scattered places in his notebooks, Ramanujan recorded some theorems for calculating singular moduli and also recorded several values of singular moduli. In this paper, we establish several general theorems for the explicit evaluations of singular moduli. We also obtain some values of class invariants and singular moduli.

2000 Mathematics Subject Classification: Primary 33D10, 11F27
Key words and phrases: Singular Moduli, Modular equation, Class invariant.

1 Introduction

In Chapter 16, of his second notebook \cite{2}, \cite{10} Ramanujan has defined his theta-function as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$
Following Ramanujan, we define
\[ \chi(q) := (-q; q^2)_\infty, \]
where
\[ (a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1. \]

The ordinary hypergeometric series \( _2F_1(a, b; c; x) \) is defined by
\[ _2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \]
where
\[ (a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), \text{ for } n \geq 1, |x| < 1. \]

The complete elliptic integral of the first kind \( K := K(\alpha) \) associated with the modulus \( \alpha, 0 < \alpha < 1, \) is defined by
\[ K(\alpha) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \alpha \sin^2 \phi}} = \frac{\pi}{2} _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha \right) \]
where the latter representation is achieved by expanding the integrand in a binomial series and integrating termwise. Singular modulus \( \alpha := \alpha_n \) is that unique positive number between 0 and 1 satisfying
\[ (1.1) \sqrt{n} = \frac{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha_n \right)}{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \alpha_n \right)}; \quad n \in \mathbb{Q}^+. \]

At scattered places in his first notebook, Ramanujan recorded several values of singular moduli \( \alpha_n := \alpha(e^{-\pi\sqrt{n}}) \) in terms of units. On page 82 of his first notebook, he gave some theorems for calculating \( \alpha_n \) when \( n \) is even. J. M. Borwein and P. B. Borwein \([6]\) had calculated some of Ramanujan’s values for \( \alpha_n \). Watson \([12]\) used the formula found in Ramanujan’s first notebook \([10\text{, vol.1, p. } 320]\), to prove the value of \( k_{210} \), where \( \alpha_n = k_n^2, \) which Ramanujan wrote in his second letter to Hardy \([11\text{ p. xxix}]\).
Let 
\[ Z(r) := Z(r; x) = \binom{1}{1/2, 1/2 - 1/2}^{2F1} r \cdot \frac{r - 1}{r}; 1; x \] 
and 
\[ q_r := q_r(x) := exp \left( -\pi \csc \left( \frac{\pi}{r} \right) \frac{2F1(\frac{1}{2}, \frac{r - 1}{2}; 1; 1 - x)}{2F1(\frac{1}{2}, \frac{r - 1}{2}; 1; x)} \right) \]
where \( r = 2, 3, 4, 6 \) and \( 0 < x < 1 \).

Let \( n \) denote a fixed natural number, and assume that
\[ n \frac{2F1(\frac{1}{r}, \frac{r - 1}{r}; 1; 1 - \alpha)}{2F1(\frac{1}{r}, \frac{r - 1}{r}; 1; \alpha)} = \frac{2F1(\frac{1}{r}, \frac{r - 1}{r}; 1; 1 - \beta)}{2F1(\frac{1}{r}, \frac{r - 1}{r}; 1; \beta)}, \]
where \( r = 2, 3, 4, \) or \( 6 \). Then a modular equation of degree \( n \) in theory of elliptic functions of signature \( r \) is a relation between moduli \( \alpha \) and \( \beta \) induced by (1.2). We often say that \( \beta \) is of degree \( n \) over \( \alpha \) and \( m(r) := \frac{Z(r : \alpha)}{Z(r : \beta)} \) is called multiplier. We also use the notations \( Z_1 := Z_1(r) = Z(r : \alpha) \) and \( Z_n := Z_n(r) = Z(r : \beta) \) to indicate that \( \beta \) has degree \( n \) over \( \alpha \). When the context is clear, we omit the argument \( r \) in \( q_r, Z(r) \) and \( m(r) \).

Ramanujan class invariants are defined as
\[ G_n := 2^{-1/4} q^{-1/24} \chi(q) = \{4\alpha(1 - \alpha)\}^{-1/24} \]
and
\[ g_n := 2^{-1/4} q^{-1/24} \chi(-q) = \{4\alpha(1 - \alpha)^{-2}\}^{-1/24}. \]

On pages 294–299, Ramanujan recorded table of values for 77 class invariants or monic irreducible polynomials. In [13], [14], Watson proved 24 of Ramanujan’s class invariants from Ramanujan’s paper [10]. Watson also wrote further four papers [15], [16], [17], [18] on the calculation of class invariants. In [4], B. C. Berndt, H. H. Chan and L. -C. Zhang has used class field theory, Galois theory and Kronecker’s limit formula to justify Watson’s assumptions and calculated some values of \( G_n \). In [1], N. D. Baruah has established the value of \( G_{217} \). In [9], Mahadeva Naika and K. Sushan Baiy
have established several new explicit evaluations of class invariants and singular moduli. In [8], Mahadeva Naika has established several new explicit values for $G_n$.

In this paper, we establish several general formulas for explicit evaluations of singular moduli using Ramanujan’s modular equations. We also obtain some values of class invariant and singular moduli.

2 Preliminary Results

In this section, we collect the identities which are useful in proving our main results.

Lemma 2.1. If $\beta$ is of degree 3 over $\alpha$, then

\[(2.1) \quad (\alpha \beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} = 1.\]

For a proof of Lemma 2.1 see Entry 5 of Chapter 19 in [2, pp. 230–231].

Lemma 2.2. If $\beta$ is of degree 5 over $\alpha$, then

\[(2.2) \quad (\alpha \beta)^{1/2} + \{(1 - \alpha)(1 - \beta)\}^{1/2} + 2 \{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6} = 1.\]

For a proof of Lemma 2.2 see Entry 13 of Chapter 19 in [2, pp. 280–282].

Lemma 2.3. If $\beta$ is of degree 7 over $\alpha$, then

\[(2.3) \quad (\alpha \beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} = 1.\]

For a proof of Lemma 2.3 see Entry 19 of Chapter 19 in [2, pp. 314–315].

Lemma 2.4. If $\beta$ is of degree 11 over $\alpha$, then

\[(2.4) \quad (\alpha \beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} + 2 \{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/12} = 1.\]

For a proof of Lemma 2.4 see Entry 7 of Chapter 20 in [2, p. 363].
Lemma 2.5. If $\beta$ is of degree 23 over $\alpha$, then
\[(2.5) \ (\alpha \beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} + 2^{2/3} \{\alpha \beta(1 - \alpha)(1 - \beta)\}^{1/24} = 1.\]
For a proof of Lemma 2.5 see Entry 15 of Chapter 20 in [2, p. 411].

Lemma 2.6. We have
\[(2.6) \ 2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}.\]
Using (1.4), we obtain (2.6).

Lemma 2.7. If $a^2 - qb^2 = d^2$, a perfect square, then
\[(2.7) \ \sqrt{a + b\sqrt{q}} = \sqrt{\frac{a + d}{2}} + (\text{sgn } b)\sqrt{\frac{a - d}{2}}.\]
The above lemma is due to Bruce Reznick. For a proof of Lemma 2.7 via Chebyshev polynomials, see [3] p. 150.

Lemma 2.8. We have
\[(2.8) \ 2\sqrt{2} \left[ \frac{g_n^3 g_{9n}^3}{g_n^3 g_{9n}^3} + \frac{1}{g_n^3 g_{9n}^3} \right] = \frac{g_{6n}^6}{g_n^6} - \frac{g_{6n}^6}{g_{9n}^6},\]
\[(2.9) \ 2\sqrt{2} \left[ \frac{G_n^3 G_{9n}^3}{G_n^3 G_{9n}^3} - \frac{1}{G_n^3 G_{9n}^3} \right] = \frac{G_{6n}^6}{G_n^6} + \frac{G_{6n}^6}{G_{9n}^6},\]
\[(2.10) \ 2 \left[ \frac{G_n^2 G_{25n}^2}{G_n^2 G_{25n}^2} - \frac{1}{G_n^2 G_{25n}^2} \right] = \frac{G_{25n}^3}{G_n^3} + \frac{G_{25n}^3}{G_{25n}^3}.\]

Lemma 2.9. We have
\[(2.11) \ G_n^2 G_{9n}^2 = \left( \frac{g_n^2 g_{9n}^2}{2} + \sqrt{g_n^4 g_{9n}^4 + 2g_n^{-2} g_{9n}^{-2}} \right),\]
\[(2.12) \ g_n^2 g_{9n}^2 = \left( \frac{G_n^2 G_{9n}^2}{2} + \sqrt{G_n^4 G_{9n}^4 - 2G_n^{-2} G_{9n}^{-2}} \right),\]
\[(2.13) \ g_n^2 g_{9n}^2 = \left( \frac{g_n^2 g_{9n}^2}{2} + \sqrt{g_n^4 g_{9n}^4 + 2g_n^{-2} g_{9n}^{-2}} \right),\]
\[(2.14) \ G_n^2 G_{9n}^2 = \left( \frac{G_n^2 G_{9n}^2}{2} + \sqrt{G_n^4 G_{9n}^4 - 2G_n^{-2} G_{9n}^{-2}} \right).\]
Proofs can be found in [7].

# 3 Main Theorems

In this section, we obtain several general formulas connecting singular moduli and class invariants.

**Theorem 3.1.** We have

\[
\alpha_n = 2G_n^{12} \left[ \left( \frac{G_n^{12} + 1}{8} \right)^{1/2} + \left( \frac{G_n^{12} + 1}{8} - \frac{1}{2} \right)^{1/2} \right]^4 \\
\times \left[ \frac{1 + \sqrt{2}u^3}{4} - \frac{1 - \sqrt{2}u^3}{4} \right]^8,
\]

\[
= \left( -g_n^{12} + \sqrt{g_n^{24} + 1} \right)^{-2} \left[ \frac{2 + v^6}{2} - \frac{v^6}{2} \right]^8,
\]

where \( u = (G_n G_{9n})^{-1} \) and \( v = g_n g_{9n} \).

**Proof of (3.1).** Using (1.3) in (2.1), we find that

\[
\alpha \beta = \left[ 1 - \sqrt{1 - 2u^6} \right]^4.
\]

Using (1.3) and (2.7) in the above equation (3.3), we obtain the required result.

**Proof of (3.2).** Using (1.4) in (2.1), we find that

\[
\alpha \beta = \left[ -v^3 + \sqrt{v^6 + 2} \right]^8.
\]

Using (1.4) and (2.7) in the above equation (3.4), we obtain the required result.
Corollary 3.1. We have
\[(3.5) \quad \alpha_{27} = 2^{-14/3} \left(2 + \sqrt{3}\right) \left(1 - \sqrt[3]{4} - 1\right).\]

Proof. Using \(n = 3\) in (3.1) with
\[(3.6) \quad G_3 = 2^{1/12},\]
\[(3.7) \quad G_{27} = 2^{1/12} \left(\sqrt{2} - 1\right)^{-1/3},\]
we obtain the required result.

Theorem 3.2. We have
\[(3.8) \quad \alpha_{25n} = 2G_{n}^{12} \left[\sqrt[4]{\frac{G_{n}^{12} + 1}{8}} + \frac{1}{2} + \sqrt[4]{\frac{G_{n}^{12} + 1}{8} - \frac{1}{2}}\right]^4 \times \left[\sqrt[4]{\frac{1 - 2u^4 + u^6}{4}} - \sqrt[4]{\frac{1 - 2u^4 - u^6}{4}}\right]^4,
\]
where \(u = (G_nG_{25n})^{-1}\).

Proof. Using (1.3) in (2.2), we deduce that
\[(3.9) \quad \alpha\beta = \left[\frac{(1 - 2u^4) - \sqrt{(1 - 2u^4)^2 - u^{12}}}{2}\right]^2.\]
Using (1.3) and (2.7) in (3.9), we deduce the required result.

Corollary 3.2. We have
\[(3.10) \quad \alpha_{45} = \frac{1}{2} + \left(255\sqrt{5} - 570\right)^{1/2} \left(10\sqrt{3} + 17 - 8\sqrt{5} - \frac{68}{\sqrt{15}}\right).\]

Proof. From [3 p.191], we have
\[(3.11) \quad G_{45} = \left(2 + \sqrt{5}\right)^{1/4} \left(\frac{\sqrt{3} + \sqrt{5}}{\sqrt{2}}\right)^{1/3}.\]
Using (3.11) with $n = \frac{9}{5}$ in (2.10), we deduce that

\[(3.12) \quad G_{\frac{9}{5}} = \left(2 + \sqrt{5}\right)^{1/4} \left(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{2}}\right)^{1/3}.
\]

Using (3.11) and (3.12) with $n = \frac{9}{5}$ in (3.8), we obtain the required result.

**Remark:** A different proof of $\alpha_{45}$ can be found in [3, p. 290].

**Theorem 3.3.** We have

\[(3.13) \quad \alpha_{49n} = 2G_n^{12} \left[\sqrt{G_n^{12} + \frac{1}{8}} + \frac{1}{2} + \sqrt{G_n^{12} + \frac{1}{8}} - \frac{1}{2}\right]^4 \times \left[1 - \sqrt{1 - 2\sqrt{2}u^3}\right]^8,\]

\[(3.14) \quad = \left(-g_n^{12} + \sqrt{g_n^{24} + 1}\right)^{-2} \left[\sqrt{\frac{2\sqrt{2} + v^3}{2\sqrt{2}} - \sqrt{\frac{v^3}{2\sqrt{2}}}}\right]^{16},\]

where $u = (G_nG_{49n})^{-1}$ and $v = g_ng_{49n}$.

**Proof of (3.13).** Using (1.3) in (2.3), we find that

\[(3.15) \quad \alpha\beta = \left[1 - \sqrt{1 - 2\sqrt{2}u^3}\right]^8.\]

Using (1.3) and (2.7) in equation (3.15), we obtain the required result (3.13).

**Proof of (3.14).** Using (1.4) and (2.7) in (2.3), we find that

\[(3.16) \quad \alpha\beta = \left[\sqrt{\frac{2\sqrt{2} + v^3}{2\sqrt{2}} - \sqrt{\frac{v^3}{2\sqrt{2}}}}\right]^{16}.\]

Using (1.4) and (2.7) in the above equation (3.16), we obtain the required result (3.14).
Corollary 3.3. We have

\[ \alpha_{49} = 2 \left( \frac{1 - \sqrt{-7 - 2\sqrt{7} + \sqrt{2\sqrt{343} + 3\sqrt{2\sqrt{7}}}}}{2} \right)^8. \]

**Proof.** From [3] p. 191, we have

\[ G_{49} = \left( \frac{7^{1/4} + \sqrt{4 + \sqrt{7}}}{2} \right). \]

Using \( n = 1 \) with (3.18) in (3.13), we obtain the required result.

Corollary 3.4. We have

\[ \alpha_{14} = \left( \sqrt{\frac{2\sqrt{2} + 1}{2\sqrt{2}} - \sqrt{\frac{1}{2\sqrt{2}}} \right)^{16} \]

\[ \times \left[ \sqrt{498 + 352\sqrt{2} - 4\sqrt{30926 + 21868\sqrt{2}}} \right. \]

\[ + \sqrt{497 + 352\sqrt{2} - 4\sqrt{30926 + 21868\sqrt{2}}} \right]^2. \]

**Proof.** From [3] p. 200, we have

\[ g_{14} = \sqrt{\frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2}}. \]

Using \( n = \frac{2}{7} \) in (2.8), we deduce that

\[ g_{\frac{2}{7}} = \sqrt{\frac{1 + \sqrt{2} - \sqrt{2\sqrt{2} - 1}}{2}}. \]

Using (3.20) and (3.21) with \( n = \frac{2}{7} \) in (3.14), we obtain the required result.
Theorem 3.4. We have

\[
(3.22) \alpha_{121n} = 2G_n^{12} \left[ \sqrt{\frac{G_n^{12}+1}{8}} + \frac{1}{2} + \sqrt{\frac{G_n^{12}+1}{8}} - \frac{1}{2} \right]^4 \times \left[ \sqrt{\frac{1-2u^2+\sqrt{2}u^3}{4}} - \sqrt{\frac{1-2u^2-\sqrt{2}u^3}{4}} \right]^8,
\]

\[
(3.23) = \left( -g_n^{12} + \sqrt{g_n^{24}+1} \right)^{-2} \left[ -v(v^2+2) + \frac{v^2(v^2+2)^2+2}{\sqrt{2}} \right]^8,
\]

where \( u = (G_n G_{121n})^{-1} \) and \( v = g_n g_{121n} \).

Proof of (3.22). Using (1.3) and (2.7) in (2.4), we deduce that

\[
(3.24) \alpha \beta = \left[ \sqrt{\frac{1-2u^2+\sqrt{2}u^3}{4}} - \sqrt{\frac{1-2u^2-\sqrt{2}u^3}{4}} \right]^8.
\]

Using (1.3) and (2.7) in (3.24), we deduce the required result (3.22).

Proof of (3.23). Using (1.4) in (2.4), we deduce that

\[
(3.25) \alpha \beta = \left[ -v(v^2+2) + \frac{v^2(v^2+2)^2+2}{\sqrt{2}} \right]^8.
\]

Using (1.4) in the above equation (3.25), we find the required result (3.23).

Theorem 3.5. We have

\[
(3.26) \alpha_{529n} = 2G_n^{12} \left[ \sqrt{\frac{G_n^{12}+1}{8}} + \frac{1}{2} + \sqrt{\frac{G_n^{12}+1}{8}} - \frac{1}{2} \right]^4 \times \left[ \frac{(1-\sqrt{2}u) + \sqrt{(1-\sqrt{2}u)^2 - 2\sqrt{2}u^3}}{2} \right]^8,
\]

\[
(3.27) = \left( -g_n^{12} + \sqrt{g_n^{24}+1} \right)^{-2} \times \left[ \frac{-\sqrt{2}\sqrt{v+\sqrt{2}} + \sqrt{2\sqrt{v+\sqrt{2}}^2+4}}{2} \right]^{16},
\]
Some theorems on the explicit ... 81

where \( u = (G_n G_{529n})^{-1} \) and \( v = g_n g_{529n} \).

**Proof of** (3.26). Using (1.3) in (2.5), we deduce that

\[
\alpha \beta = \left( \frac{1 - \sqrt{2}u}{2} + \sqrt{1 - \sqrt{2}u} \right)^8.
\]

Using (1.3) and (2.7) in (3.28), we obtain the required result (3.26).

**Proof of** (3.27). Using (1.4) in (2.5), we deduce that

\[
\alpha \beta = \left( -\frac{\sqrt{2}v}{2} + \frac{v + \sqrt{2}}{2} \right)^8.
\]

Using (1.4) in the above equation (3.29), we find the required result (3.27).

### 4 Explicit Evaluations of \( g_n \)

In this section, we obtain some explicit evaluations of \( g_n \).

**Theorem 4.1.** We have

\[
g_{24} = \left( \sqrt{3} + 1 \right)^{1/4} \left( \frac{\sqrt{45 + 27\sqrt{3}}}{2} + \frac{\sqrt{43 + 27\sqrt{3}}}{2} \right)^{1/12},
\]

\[
g_{83} = \left( \sqrt{3} + 1 \right)^{1/4} \left( \frac{\sqrt{45 + 27\sqrt{3}}}{2} - \frac{\sqrt{43 + 27\sqrt{3}}}{2} \right)^{1/12}.
\]

**Proof.** In [7, Th.4.5], we have

\[
g_{24} g_{24} = \sqrt{3} + 1.
\]

Using (4.3) in (2.8), we deduce that

\[
\frac{g_{24}^6}{g_{24}^6} = \sqrt{\frac{45 + 27\sqrt{3}}{2}} + \sqrt{\frac{43 + 27\sqrt{3}}{2}}.
\]

From (4.3) and (4.4), we obtain the required results.
Theorem 4.2. We have

\[
g_{96} = \left(1 + \sqrt{3}\right)^{1/4} \left(1 + \sqrt{3} + \sqrt{3 + 3\sqrt{3}}\right)^{1/4} \left(\sqrt[4]{\frac{7110 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}} + \sqrt[4]{\frac{7106 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}}\right)^{1/12},
\]

(4.5)

\[
g_{32} = \frac{1}{3} \left(1 + \sqrt{3}\right)^{1/4} \left(1 + \sqrt{3} + \sqrt{3 + 3\sqrt{3}}\right)^{1/4} \left(\sqrt[4]{\frac{7110 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}} - \sqrt[4]{\frac{7106 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}}\right)^{1/12},
\]

(4.6)

\[
g_{240} = 2^{1/8} \left(\sqrt{3} + 1\right)^{1/4} \left(\sqrt{10} + 3\right)^{1/4} \left(\sqrt{6} + \sqrt{5}\right)^{1/4} \left(\sqrt[4]{\frac{3513510\sqrt{10} + 6410880\sqrt{3} + 11104056 + 2028510\sqrt{30}}{16}} + \sqrt[4]{\frac{3513510\sqrt{10} + 6410880\sqrt{3} + 11104040 + 2028510\sqrt{30}}{16}}\right)^{1/12},
\]

(4.7)
Some theorems on the explicit ... 83

\[ g_{w} = 2^{1/8} \left( \sqrt{3} + 1 \right)^{1/4} \left( \sqrt{10} + 3 \right)^{1/4} \left( \sqrt{6} + \sqrt{5} \right)^{1/4} \]

(4.8) \[ \left( \sqrt[4]{\frac{3513510 \sqrt{10} + 6410880 \sqrt{3} + 11104056 + 2028510 \sqrt{30}}{16}} \right) ^{1/12} \]

Proof of (4.5). Using (4.3) in (2.13), we obtain the required result.

Since the proofs of (4.6) – (4.8) are similar to the proof of (4.5), we omit the details.

5 Explicit Evaluations of \( G_n \)

In this section, we obtain some explicit evaluations of \( G_n \).

Theorem 5.1. We have

\[ G_{24} = \left( \frac{1 + \sqrt{3} + \sqrt[4]{3 + 3\sqrt{3}}}{2} \right)^{1/4} \left( \sqrt[12]{\frac{45 + 27\sqrt{3}}{2}} - \sqrt[12]{\frac{43 + 27\sqrt{3}}{2}} \right) \]

(5.1) \[ \left( \sqrt[4]{\frac{7110 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}} + \sqrt[4]{\frac{7106 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}} \right)^{1/12} \]
84

K. Sushan Bairy

\[ G_{\frac{2}{3}} = \left( \frac{1 + \sqrt{3} + \sqrt{3 + 3\sqrt{3}}}{2} \right)^{1/4} \left( \sqrt{\frac{45 + 27\sqrt{3}}{2}} + \sqrt{\frac{43 + 27\sqrt{3}}{2}} \right)^{1/12} \]

(5.2)

\[
\left( \sqrt{\frac{7110 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}} \right)^{1/12} - \left( \sqrt{\frac{7106 + 4104\sqrt{3} + 27\sqrt{141706 + 81814\sqrt{3}}}{4}} \right)^{1/12}.
\]

Theorem (5.1) is obtained by using Theorem (4.1) and the formula \( g_{4n} = 2^{i} G_{n} g_{n} \).

6 Explicit Evaluations of \( \alpha_{n} \)

In this section, we obtain some explicit evaluations of \( \alpha_{n} \).

**Theorem 6.1.** We have

(6.1)

\[
\alpha_{24} = \left( 69 + 40\sqrt{3} - 28\sqrt{6} - 48\sqrt{2} + 6\sqrt{256 + 153\sqrt{3} - 2\sqrt{2333 + 1347\sqrt{3}}} \right)^{2} \times \left( \sqrt{6 + 3\sqrt{3}} - \sqrt{5 + 3\sqrt{3}} \right)^{8},
\]

(6.2)

\[
\alpha_{\frac{8}{3}} = \left( 69 + 40\sqrt{3} - 28\sqrt{6} - 48\sqrt{2} + 6\sqrt{256 + 153\sqrt{3} - 2\sqrt{2333 + 1347\sqrt{3}}} \right)^{-2},
\]
(6.3)\[
\alpha_{39} = 2^4 \left( \frac{\sqrt{13} + 3}{2} \right)^2 \left( \frac{5 + \sqrt{13}}{8} - \frac{\sqrt{13} - 3}{8} \right)^{12} \left( \frac{1}{2} - \frac{1}{4} \frac{-3 + 3\sqrt{13}}{2} \right)^4 \times \left( \sqrt{1295056 + 359184\sqrt{13}} - 48\sqrt{1455864558 + 403784178\sqrt{13}} \right. \\
+ \left. \sqrt{1295055 + 359184\sqrt{13}} - 48\sqrt{1455864558 + 403784178\sqrt{13}} \right),
\]

(6.4)\[
\alpha_{13} = 2^{-4} \left( \frac{\sqrt{13} - 3}{2} \right)^2 \left( \frac{5 + \sqrt{13}}{8} + \frac{\sqrt{13} - 3}{8} \right)^{12} \times \left( \sqrt{1295056 + 359184\sqrt{13}} - 48\sqrt{1455864558 + 403784178\sqrt{13}} \right.
\]
\[- \left. \sqrt{1295055 + 359184\sqrt{13}} - 48\sqrt{1455864558 + 403784178\sqrt{13}} \right).
\]

Proofs are similar to the proof of corollary 3.1, so we omit the details.

Acknowledgement

I wish to thank Dr. M. S. Mahadeva Naika for his valuable suggestions and encouragement during the preparation of this paper. I also wish to thank the referee for their valuable suggestions.

References


Department of Mathematics,
Central College Campus,
Bangalore University,
Bangalore-560 001, INDIA
E-mail: ksbairy@rediffmail.com