New results in discrete asymptotic analysis

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Abstract

We present the order of magnitude of the sequence \( \Omega_{n,r}^{[\alpha\beta]} \) of general term \( \Omega_{n,r}^{[\alpha\beta]} = \frac{\alpha(\alpha + r)(\alpha + 2r)\ldots(\alpha + (n-1)r)}{\beta(\beta + r)(\beta + 2r)\ldots(\beta + (n-1)r)} \), where \( r > 1 \) and \( 0 < \alpha < \beta \leq r \) are fixed.

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1 Introduction

The asymptotic analysis, usually considered in connection with the functions of real or complex variable, can be also applied to the study of the functions of natural variable \( n \), for \( n \to \infty \).

This study forms the discrete asymptotic analysis. The principal purposes of this analysis are to obtain, for a given sequence:

(a) the order of magnitude;
(β) the convergence of the given sequence or of a derived one;
(γ) the first iterated limit (respecting an auxiliary scale of sequences);
(γ′) a two sided estimation for the convergence; it must permit to find
again the limit of (γ);
(δ) (if possible) the asymptotic expansion (respecting the given scale of
sequences).

Some examples are classic.

For (α). The harmonic sum \( H_n = 1 + 1/2 + 1/3 + \ldots + 1/n \) has the
order of magnitude of \( \ln n + \gamma \), namely
\[
H_n = \ln n + \gamma + \varepsilon_n,
\]
where \( \gamma = \lim_{n \to \infty} (H_n - \ln n) = 0,577\ldots \) is the famous constant of \( Euler \) and
\( \varepsilon_n \to 0 \).

The factorial’s magnitude is described by the formula of \( Stirling \)
\( n! \approx n^n e^{-n} \sqrt{2\pi n} \), having the precise signification that
\[
\lim_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.
\]

The number \( \pi(n) \) of primes not exceeding a given number \( n \) is
\( \pi(n) \approx n/(\ln n) \) (i.e. \( \lim_{n \to \infty} (\pi(n))/(n/(\ln n)) = 1 \)). This formula has a rich
history related to \( Legendre, Gauss, Tchebycheff, Hadamard, De La Vallée, 
Poussin \) and others.

For (β). The sequence \((e_n)_n\) of general term \( e_n = (1 + 1/n)^n \) defines by
its limit, the famous constant \( e \) of \( Napier \) and \( Euler \).

For \((H_n)_n\) the limit is \( \infty \), but \( \gamma_n = H_n - \ln n \) is an convergent sequence
related to \( H_n \); its limit defines the constant of \( Euler \).
If we consider $S_n = \log_2 3 + \log_3 4 + \ldots + \log_n(n + 1)$ (a sum of L. Panaitopol), then the sequence $x_n = S_n - (n - 1) - \ln(\ln n)$ is convergent to $x = \gamma + \lim_{n \to \infty} \left( \sum_{k=2}^{n} \frac{1}{k \ln k} - \ln \ln n \right)$.

For $(\gamma)$. Two examples of first iterated limits are the following

$$\lim_{n \to \infty} n \left( e - \left(1 + \frac{1}{n}\right)^n \right) = \frac{e}{2};$$

$$\lim_{n \to \infty} n (\gamma_n - \gamma) = \frac{1}{2}.$$

For $(\gamma')$. The corresponding two sided estimations are

$$\frac{e}{2n + 2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n + 1},$$

$$\frac{1}{2n + 1} < \gamma_n - \gamma < \frac{1}{2n}.$$

For $(\delta)$. As examples of asymptotic developments (expansions) we can consider the corresponding for $H_n$ and $n!$ and others.

Many examples are known.

In the following we will use some of standard notations.

- $a_n = O(b_n)$ if there are two constants $M > 0$ and $c > 0$ such that $|a_n| < M|b_n|$ for any $n \in \mathbb{N}$, $n > c$.
- $a_n = o(b_n)$ if $\lim_{n \to \infty} a_n/b_n = 0$.
- $O(1)$ is a notation for a sequence which is bounded.
- $o(1)$ is a notation for a sequence which tends to zero, where $n \to \infty$.
- the sequences $(a_n)_n$ and $(b_n)_n$ are called asymptotic equivalent and we write $a_n \sim b_n$ if $\lim_{n \to \infty} a_n/b_n = 1$. 
2 The starting example

Let

\[ \Omega_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n} \]

be. This expression has a certain importance, because it appears often in many concrete questions of analysis:

- It is related to the bigger binomial coefficient (the middle term) of \((1 + 1)^n\), namely \(\binom{2n}{n} = 4^n \Omega_n\).
- It is related to the Mac Laurin expansions of \((1 + x)^{1/2}\), \((1 - x)^{1/2}\), \((1 - x^2)^{1/2}\) and arcsin \(x\).
- It is related to the so called integrals of Wallis, \(I_n = \int_0^\frac{\pi}{2} \sin^n x \, dx\).
- It is related to the formula of Wallis

\[ \lim_{n \to \infty} W_n = \frac{\pi}{2}, \]

where

\[ W_n = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n - 1)(2n + 1)}. \]

because of the relation

\[ W_n = \frac{1}{\Omega_n^2} \frac{1}{2n + 1}. \]

Just because of these, the expression \(\Omega_n\) was intensively studied.

Firstly, from the inequality

\[ 0 < \Omega_n < \frac{1}{\sqrt{2n + 1}} \]
it results \( \lim_{n \to \infty} \Omega_n = 0 \).

From (2) and (4) we obtain \( \lim_{n \to \infty} \Omega_n \sqrt{n} = 1/\sqrt{\pi} \), i.e.

\[
(6) \quad \Omega_n = O \left( \frac{1}{\sqrt{\pi n}} \right).
\]

A two sided estimation of \( \Omega_n \) (more accurate than (5)), namely

\[
(7) \quad \frac{1}{\sqrt{\pi \left(n + \frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left(n + \frac{1}{4}\right)}},
\]

is called in a famous book of D.S. Mitrinović and P. M. Vasić [5] ”the inequality of Wallis”.

It was refined by D. N. Kazarinoff in 1956 (see [1])

\[
(8) \quad \frac{1}{\sqrt{\pi \left(n + \frac{1}{4}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left(n + \frac{1}{4n}\right)}},
\]

respectively L. Panaitopol in 1985 (see [6])

\[
(9) \quad \frac{1}{\sqrt{\pi \left(n + \frac{1}{4n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi \left(n + \frac{1}{4n}\right)}},
\]

and later we have given its asymptotic expansion

\[
(10) \quad \Omega_n = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n + \frac{1}{4} + \frac{1}{32n} - \frac{1}{128n^2} + \ldots}}
\]

(see [7], [8]).
3 The expressions considered by us

Let $r \in \mathbb{N}^*$, $r > 1$ be. Also let $\alpha$ and $\beta$ be the real numbers, such that $0 < \alpha < \beta \leq r$. Consider the sequence $(\Omega_{n,r}^{\alpha,\beta})_{n \geq 1}$ with general term defined by the equality

$$\Omega_{n,r}^{\alpha,\beta} \overset{\text{def}}{=} \frac{\alpha(\alpha + r)(\alpha + 2r) \ldots (\alpha + (n-1)r)}{\beta(\beta + r)(\beta + 2r) \ldots (\beta + (n-1)r)}.$$ 

This generalizes $\Omega_n$, which can be obtained for $r = 2$, $\alpha = 1$ and $\beta = 2$. We are interested to obtain the order of magnitude of $\Omega_{n,r}^{\alpha,\beta}$.

4 The order of magnitude

We have

$$\Omega_{n,r}^{\alpha,\beta} = \frac{r^n \frac{\alpha}{r} \left( \frac{\alpha}{r} + 1 \right) \left( \frac{\alpha}{r} + 2 \right) \ldots \left( \frac{\alpha}{r} + (n-1) \right)}{r^n \frac{\beta}{r} \left( \frac{\beta}{r} + 1 \right) \left( \frac{\beta}{r} + 2 \right) \ldots \left( \frac{\beta}{r} + (n-1) \right)}.$$ 

From the formula $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we obtain by iteration

$$\Gamma(x + p + 1) = x(x+1)(x+2) \ldots (x+p)\Gamma(x)$$

($p \in \mathbb{N}, x, x + p \notin \{-1, -2, -3, \ldots\}$) i.e.

$$x(x+1)(x+2) \ldots (x+p) = \frac{\Gamma(x + p + 1)}{\Gamma(x)}.$$ 

Applying (12) two times in (11) we obtain

$$\Omega_{n,r}^{\alpha,\beta} = \frac{\Gamma \left( \frac{\beta}{r} \right)}{\Gamma \left( \frac{\alpha}{r} \right)} \cdot \frac{\Gamma \left( \frac{\alpha}{r} + n \right)}{\Gamma \left( \frac{\beta}{r} + n \right)},$$

(13)
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which is a first expression of \( \Omega^{[a]}_{n,r} \).

To obtain the order of magnitude of \( \Omega^{[b]}_{n,r} \), we will use the Stirling approximation of \( \Gamma \), namely

\[
\Gamma(x + 1) \sim x^x e^{-x \sqrt{2\pi x}} \quad (x > 0).
\]

So we obtain

\[
\Omega^{[b]}_{n,r} \sim \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \frac{(n + \frac{\alpha}{r} - 1)^{n+\frac{\beta}{r}-1} e^{-\left(n+\frac{\beta}{r}-1\right)} \sqrt{2\pi \left(n + \frac{\alpha}{r} - 1\right)}}{(n + \frac{\beta}{r} - 1)^{n+\frac{\alpha}{r}-1} e^{-\left(n+\frac{\alpha}{r}-1\right)} \sqrt{2\pi \left(n + \frac{\beta}{r} - 1\right)}}.
\]

i.e.

\[
\Omega^{[b]}_{n,r} \sim \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \frac{(n + \frac{\alpha}{r} - 1)^{n+\frac{\beta}{r}-1}}{(n + \frac{\beta}{r} - 1)^{n+\frac{\alpha}{r}-1}} \cdot \frac{1}{\left(n + \frac{\beta}{r} - 1\right)^{\frac{\beta - \alpha}{2}}} \cdot \frac{1}{e^{\frac{\beta - \alpha}{r}}}
\]

Because of the equality

\[
\lim_{n \to \infty} \left(\frac{n + \frac{\alpha}{r} - 1}{n + \frac{\beta}{r} - 1}\right)^{n+\frac{\beta}{r}-1} = e^{\frac{\alpha - \beta}{r}},
\]

we obtain

\[
\lim_{n \to \infty} \frac{\Omega^{[b]}_{n,r}}{n^{\frac{\beta - \alpha}{r}}} = \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)}.
\]
This conducts us to find the magnitude of $\Omega_{n,r}^{(\alpha,\beta)}$, namely we have obtained the

**Theorem 1** We have

\[
\Omega_{n,r}^{(\alpha,\beta)} \sim \frac{\Gamma\left(\frac{\beta}{r}\right)}{\Gamma\left(\frac{\alpha}{r}\right)} \cdot \frac{1}{n^{\frac{\beta}{\alpha}}}
\]

The proof has given before.

In the case of $\Omega_n$ ($r = 2$, $\alpha = 1$ and $\beta = 2$) the formula (13) gives

\[
\Omega_n = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n + 1)}
\]

The first author remembers a view in a first time of the formula (16) starting directly from the definition of $\Omega_n$, in a conversation with the regretted Professor Alexandru Lupas. This conducted us to two joint papers [3], [4] and stimulated us to consider and study the general case of $\Omega_{n,r}^{(\alpha,\beta)}$.

The formula (15) becomes

\[
\Omega_n \sim \frac{1}{\sqrt{\pi n}}
\]

finding again (6).

The presence of $\sqrt{n}$ has now a natural explanation by our overview. The apparition of $\sqrt{\pi}$ is related only to the well-known relation between $\Gamma(1/2)$ and $\sqrt{\pi}$. 
References


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