On an expansion theorem in the finite operator calculus of G-C Rota

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Abstract

Using a identity for linear operators we present here the Taylor formula in the umbral calculus.

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1 Introduction

We consider the algebra of all polynomials $p(t)$ in one variable over a field of characteristic zero, to be denoted $\Pi$.

We denote by $\Pi^*$ the linear space of linear operators on $\Pi$ to $\Pi$. For example $D \in \Pi^*$, $Dp(t) = p'(t)$ (the derivative), $E^a \in \Pi^*$, $(E^ap)(t) = p(t + a)$ (the shift operator), $Ip(t) = p(t)$ (the identity).

We denote by $\Pi_t^*$ the set of shift invariant operators

$$\Pi_t^* = \{ T \mid TE^a = E^aT, (\forall) a \}$$

and by $\Pi_\delta^*$ the set of delta operators

$$\Pi_\delta^* = \{ Q \in \Pi_t^* \mid Qx \text{ is a nonzero constant} \} .$$
Delta operators possess many of the properties of the derivative operator $D$. For example if $Q$ is a delta operator, then $Qa = 0$ for every constant $a$. Next, if $p(t)$ is a polynomial of degree $n$ and $Q \in \Pi^*_\delta$, then $Qp(t)$ is a polynomial of degree $n - 1$.

A polynomial sequence $(p_n(t))$ ($\deg p_n = n, \ n = 0, 1, 2, \ldots$) is called the sequence of basic polynomials for $Q \in \Pi^*_\delta$ if $p_0(t) = 1$, $p_n(0) = 0$ for $n \geq 1$ and $Qp_n(t) = np_{n-1}(t)$ for $n \geq 1$.

It is known the following theorem

**Theorem 1.** i) Every delta operator has a unique sequence of basic polynomials.

ii) If $(p_n(t))$ is a basic sequence for some delta operator $Q$ then it is a sequence of polynomials of binomial type.

iii) If $(p_n(t))$ is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator.

The following theorem generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

**Theorem 2.** For $T \in \Pi^*_t$ and $Q \in \Pi^*_\delta$ with basic set $(p_n)$, we have

\[
T = \sum_{k \geq 0} \frac{(T p_k)(0)}{k!} Q^k.
\]

We consider now an operator $X \notin \Pi^*_t$, defined by $Xp(t) = tp(t)$ and for any operator $T$ defined on $\Pi$, the operator

\[
T' = TX - XT
\]

will be called the Pincherle derivative of the operator $T$.

We observe that $D' = I$, $(E^a)' = aE^a$, $I' = O$ (the null operator).
2 The Bernoulli identity

Let $T, S$ be two linear operators such that

\[(2) \quad TS - ST = I.\]

For example $DX - XD = I$. From (2) we obtain

\[TS^2 = S^2T + 2S\]

and by induction

\[(3) \quad TS^n = S^nT + nS^{n-1}, \quad n \geq 1.\]

Starting with the identity

\[(4) \quad \sum_{k=0}^{n} (\alpha_k - \alpha_{k+1}) = \alpha_0 - \alpha_{n+1}\]

for

\[(5) \quad \alpha_0 = 0, \quad \alpha_k = \frac{(-1)^k}{(k-1)!} S^{k-1}T^k, \quad k \geq 1\]

and using (3) we get

\[\alpha_k - \alpha_{k+1} = \frac{(-1)^k}{k!} TS^kT^k\]

and hence

\[(6) \quad T \sum_{k=0}^{n} \frac{(-1)^k}{k!} S^kT^k = \frac{(-1)^n}{n!} S^nT^{n+1}.\]

This is the Bernoulli identity obtained by O.V. Viskov (see [1], [3]).
3 The main result

Let $Q$ be a delta operator with the basic set $(p_n(t))$. Hence $p_0(x) = 1$, $p_n(0) = 0$ for $n \geq 1$ and $Q p_n = np_{n-1}$ for $n \geq 1$.

**Definition 1.** We define the $Q$-integral operator as a linear operator

$$I_Q = \int dt$$

by

$$(I_Q p_n)(t) = \int p_n(t) dt = \frac{1}{n+1}p_{n+1}(t),$$

for $n \geq 0$. We denote

$$(7) \quad \int_{\alpha}^{x} (Qp)(t) dt = p(x) - p(\alpha).$$

**Definition 2.** We define next the pseudo $Q$-integral operator

$$T_Q \in \Pi^*, (T_Q p_n)(t) = p_{n+1}(t).$$

**Remark 1.** For $Q = D$ we have $p_n(t) = t^n$, $n = 0, 1, 2, \ldots$ and $T_Q = T_D = X$, $(Xp)(t) = tp(t)$.

**Theorem 3.** We have the following Taylor expansion formula

$$\left( xI - T_Q \right)^k Q^n f \right) (x) = \sum_{k=0}^{n} \frac{((xI - T_Q)^k Q^n f \right) (x)}{k!} = \sum_{k=0}^{n} \frac{((xI - T_Q)^k Q^n f \right) (x)}{k!} + \int_{\alpha}^{x} \frac{((xI - T_Q)^n Q^{n+1} f \right) (t)}{n!} dt$$

with the rest term in the Cauchy form.

**Proof.** Let $T, S$ be as below

$$T = Q, \quad S = T_Q - xI.$$
We have \((TS - ST)p_n(t) = p_n(t)\) and hence \(TS - ST = I\). After submission into (6) we get

\[
Q \sum_{k=0}^{n} \frac{(-1)^k}{k!} (TQ - xI)^k Q^k p(t) = \frac{(-1)^n}{n!} (TQ - xI)^n Q^{n+1} p(t), \quad p(t) \in \Pi.
\]

Apply \(\int_{\alpha}^{x} dt\) to both sides where, of course, \(t\) is the variable, and using (7) to obtain

\[
\sum_{k=0}^{n} \frac{((xI - TQ)^k Q^k p)}{k!} = \sum_{k=0}^{n} \frac{((xI - TQ)^k Q^{k+1} p)}{k!} + R_{n+1}(x)
\]

where the rest term \(R_{n+1}\) is in the Cauchy form

\[
R_{n+1}(x) = \int_{\alpha}^{x} \frac{((xI - TQ)^n Q^{n+1} p)(t)}{n!} dt.
\]

**Remark 2.** For \(Q = D\) we observe that \(TD = X\) and hence

\[
(TDp)(t) = tp(t).
\]

Next \(((xI - TD)p)(x) = (xp(t) - tp(t))|_{t=x} = 0\) and of (9) we obtain

\[
p(x) = \sum_{k=0}^{n} \frac{(x - \alpha)^k}{k!} p^{(k)}(\alpha) + R_{n+1}(x)
\]

with \(R_{n+1}(x) = \int_{\alpha}^{x} \frac{(x - t)^n}{n!} p^{(n+1)}(t) dt\).

**Remark 3.** We have \(DX - XD = I\) and for \(T = D\) and \(S = X\) in the Bernoulli identity we obtain

\[
D \sum_{k=0}^{n} \frac{(-1)^k}{k!} X^k D^k = \frac{(-1)^n}{n!} X^n D^{n+1}
\]

and finally a McLaurin expansion formula in the following form

\[
p(0) = \sum_{k=0}^{n} \frac{(-\alpha)^k}{k!} p^{(k)}(\alpha) + \int_{0}^{\alpha} \frac{(-t)^{n+1}}{n!} p^{(n+1)}(t) dt.
\]
References


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