A simple solution to Basel problem

Mircea Ivan

Abstract

In the following use present a simple proof of Euler’s formula

\begin{equation}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.
\end{equation}

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1 Introduction

The Basel problem is a famous problem in number theory, first posed by Pietro Mengoli in 1644, and solved by Leonhard Euler in 1735. The Basel problem asks for the precise sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).
2 The Proof

We present below a version of James D. Harper’s [4] simple proof. We use the Fubini theorem for integrals and McLaurin’s series expansion for \(\tanh^{-1}\):

\[
\frac{1}{2} \log \frac{1+y}{1-y} = \sum_{n=0}^{\infty} \frac{y^{2n+1}}{2n+1}, \quad |y| < 1.
\]

We start with the equality

\[
\text{\textbf{(2)}} \quad \int_{-1}^{1} \int_{-1}^{1} \frac{1}{1+2xy+y^2} \, dy \, dx = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{1+2xy+y^2} \, dx \, dy
\]

The left hand side of (2) gives:

\[
\int_{-1}^{1} \int_{-1}^{1} \frac{1}{1+2xy+y^2} \, dy \, dx = \int_{-1}^{1} \text{arctan} \left( \frac{x+y}{\sqrt{1-x^2}} \right) \bigg|_{y=-1}^{y=1} \, dx
\]

\[
= \int_{-1}^{1} \frac{\pi}{2\sqrt{1-x^2}} \, dx = \frac{\pi^2}{2}.
\]

The right hand side of (2) yields:

\[
\int_{-1}^{1} \int_{-1}^{1} \frac{1}{1+2xy+y^2} \, dy \, dx = \int_{-1}^{1} \log \left( 1 + \frac{2xy + y^2}{2y} \right) \bigg|_{x=-1}^{x=1} \, dy
\]

\[
= \int_{-1}^{1} \frac{\log(1+y)}{y} \, dy = 2 \int_{-1}^{1} \sum_{n=0}^{\infty} \frac{y^{2n}}{2n+1} \, dy
\]

\[
= 4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},
\]

hence,

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},
\]

which is equivalent to (1).

For readers who would enjoy seeing more proofs, see the references.
References


[3] Josef Hofbauer, *A simple proof of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ and related identities*, The American Mathematical Monthly 109(2) (Feb., 2002) 196–200.


Mircea Ivan
Technical University of Cluj-Napoca
Department of Mathematics
Str. C. Daicoviciu 15, 400020 Cluj-Napoca, Romania
e-mail: Mircea.Ivan@math.utcluj.ro