Remarks on Voronovskaya’s theorem

Heiner Gonska and Ioan Raşa

Abstract

The present note discusses various quantitative forms of Voronovskaya’s 1932 result dealing with the asymptotic behavior of the classical Bernstein operators. In particular the relationship between a result of Sikkema and van der Meer and an alternative approach of the authors is discussed.

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In a recent paper [4] the well-known theorem of Voronovskaya for the classical Bernstein operators $B_n$ was stated in the following form.

Theorem 1 For $f \in C^2[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$ one has

$$\left| n \cdot [B_n(f; x) - f(x)] - \frac{x(1-x)}{2} \cdot f''(x) \right| \leq \frac{x(1-x)}{2} \cdot \tilde{\omega} \left( f''; \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right).$$

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Here \( \tilde{\omega} \) is the least concave majorant of \( \omega \), the first order modulus of continuity, satisfying
\[
\omega(f;\epsilon) \leq \tilde{\omega}(f;\epsilon) \leq 2\omega(f;\epsilon), \epsilon \geq 0.
\]
The above inequality follows from a more general asymptotic statement which was inspired by results of Bernstein [2] and Mamedov [6]. This is given in

**Theorem 2** Let \( q \in \mathbb{N}_0, f \in C^q[0,1] \) and \( L : C[0,1] \to C[0,1] \) be a positive linear operator. Then
\[
\left| L(f;x) - \sum_{r=0}^{q} L((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right|
\leq \frac{L(|e_1 - x|^q; x)}{q!} \tilde{\omega}(f^{(q)}; \frac{L(|e_1 - x|^{q+1}; x)}{(q+1)L(|e_1 - x|^q; x)}).
\]

The following remarks are obvious:

**Remark 1** Both asymptotic statements (supposing \( L = L_n, n \in \mathbb{N} \), in Theorem 2) are in quantitative form due to the appearance of \( \tilde{\omega} \).

**Remark 2** In Theorem 1 the (absolute) moments \( L((e_1 - x)^r; x) \) and \( L(|e_1 - x|^r; x) \) are computed and/or manipulated in order to arrive at more instructive quantities. Of course this is not possible in Theorem 2 unless one makes additional assumptions on \( L \).

**Remark 3** In Theorem 1 the limit \( \frac{x(1-x)}{2} \cdot f''(x) \) is explicitly given. The inequality of Theorem 2 requires extra considerations to arrive at a comparable statement.
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Remark 4 Thinking of Theorem 2 as an asymptotic expansion (supposing again that \( L = L_n, n \in \mathbb{N} \)), this expansion is "complete" in the sense that \( q \in \mathbb{N}_0 \) is arbitrary.

In contrast to that, the expansion of Theorem 1 is "non-complete".

Remark 5 Both inequalities above do not give information about the asymptotic behaviour of quantities such as

\[ n[(B_n f)^{(k)}(x) - f^{(k)}(x)] \text{ for } k \geq 1. \]

That this is also a meaningful problem was shown in recent papers by Floater [3] and Abel and Heilmann [1], Theorem 3.3, for example.

A very interesting complete asymptotic expansion (in quantitative form) was already given some 30 years ago by Sikkema and van der Meer [8].

Theorem 3 Let \( WC^q[0,1] \) denote the set of all functions on \([0,1]\) whose \( q \)-th derivative is piecewise continuous, \( q \geq 0 \). Moreover, let \((L_n)\) be a sequence of positive linear operators \( L_n : WC^q[0,1] \rightarrow C[0,1] \) satisfying \( L_n(e_0;x) = 1 \). Then for all \( f \in fC^q[0,1], q \in \mathbb{N}_0, x \in [0,1], n \in \mathbb{N} \) and \( \delta > 0 \) one has

\[ \left| L_n(f;x) - f(x) - \sum_{r=1}^{q} \frac{L_n((e_1-x)^r;x)}{r!} \cdot f^{(r)}(x) \right| \leq c_{n,q}(x,\delta) \cdot \omega(f^{(q)};\delta). \]

Here \( c_{n,q}(x,\delta) = \delta^q \cdot L_n \left(s_{q,\mu} \left(\frac{2x^q}{\delta}\right);x\right), \)

\[ \mu = \frac{1}{2} \text{ if } L_n((e_1-x)^q;x) \geq 0, \]

\[ \mu = -\frac{1}{2} \text{ if } L_n((e_1-x)^q;x) < 0, \]
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\[ s_{q,\mu}(u) = \frac{1}{q!} \left( \frac{1}{2} \cdot |u|^q + \mu u^q \right) + \frac{1}{(q+1)!} \{ b_{q+1}(|u|) - b_{q+1}(|u| - |u|) \}. \]

\( b_{q+1} \) is the Bernoulli polynomial of degree \( q+1 \) and \( [t] = \max \{ z \in \mathbb{Z} : z \leq t \} \).

Moreover, the functions \( c_{n,q}(x, \delta) \) are best possible for each \( f \in C^q[0,1] \), \( x \in [0,1] \), \( n \in \mathbb{N} \) and \( \delta > 0 \).

In the sequel we will deal with the case \( q = 2 \) only and furthermore assume that \( L_n(e_1; x) = x \). The above theorem then implies the inequality given in

**Corollary 1**

\[ \left| L_n(f; x) - f(x) - \frac{1}{2} \cdot L_n((e_1 - x)^2; x) \cdot f''(x) \right| \leq c_{n,2}(x, \delta) \cdot \omega(f'', \delta), \]

where

\[ c_{n,2}(x; \delta) = \delta^2 \cdot L_n \left( s_{2,3} \left( \frac{e_1 - x}{\delta} \right) ; x \right) \]

\[ s_{2,3}(u) = \frac{1}{2} u^2 + \frac{1}{6} \{ b_3(|u|) - b_3(|u| - |u|) \}, \]

\[ b_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x. \]

As an alternative inequality we propose the one given in

**Theorem 4** Let \( L : C[0,1] \to C[0,1] \) be a positive linear operator satisfying \( L e_i = e_i, i = 0,1 \). Then for any \( f \in C^2[0,1], x \in [0,1] \) and \( \delta > 0 \) we have

\[ \left| L(f; x) - f(x) - \frac{1}{2} \cdot L((e_1 - x)^2; x) \cdot f''(x) \right| \]

\[ \leq \frac{1}{2} \cdot \max \left\{ L((e_1 - x)^2; x), \frac{1}{3\delta} L(|e_1 - x|^3; x) \right\} \cdot \tilde{\omega}(f''; \delta) \]

\[ \leq \max \left\{ L((e_1 - x)^2; x), \frac{1}{3\delta} \cdot L(|e_1 - x|^3; x) \right\} \cdot \omega(f'', \delta). \]
Proof Proceeding as in the considerations preceding Theorem 6.2 in [5] it can be seen that for \( f \in C^2[0, 1] \) fixed and \( g \in C^3[0, 1] \) arbitrary one gets

\[
|L(f; x) - f(x) - \frac{1}{2}L((e_1 - x)^2; x) \cdot f''(x)|
\leq L((e_1 - x)^2; x) \cdot \left\{ \frac{1}{6} \cdot \frac{L(|e_1 - x|^3; x)}{L((e_1 - x)^2; x)} \cdot \frac{2}{\delta} \cdot \delta |g'''| \right\}
\leq L((e_1 - x)^2; x) \cdot \max \left\{ 1; \frac{1}{3\delta} \cdot \frac{L(|e_1 - x|^3; x)}{L((e_1 - x)^2; x)} \right\} \cdot \left\{ \frac{1}{6} \cdot \frac{L(|e_1 - x|^3; x)}{L((e_1 - x)^2; x)} \cdot \frac{2}{\delta} \cdot \delta |g'''| \right\}.
\]

Passing to the infimum over \( g \in C^3[0, 1] \) then implies

\[
|L(f; x) - f(x) - \frac{1}{2}L((e_1 - x)^2; x) \cdot f''(x)|
\leq \max \left\{ L((e_1 - x)^2; x); \frac{1}{3\delta} \cdot L(|e_1 - x|^3; x) \right\} \cdot K \left( \frac{\delta}{2}; f''; C[0, 1], C^4[0, 1] \right)
= \frac{1}{2} \max \left\{ L((e_1 - x)^2; x); \frac{1}{3\delta} L(|e_1 - x|^3; x) \right\} \cdot \tilde{\omega}(f''; \delta).
\]

Here we used the fact that for \( f \in C[0, 1] \) and \( \delta > 0 \) one has

\[
K \left( \frac{\delta}{2}; f; C[0, 1], C^4[0, 1] \right) := \inf \left\{ ||f - g|| + \frac{\delta}{2} \cdot ||g'||; g \in C^4[0, 1] \right\} = \frac{1}{2} \tilde{\omega}(f; \delta).
\]

See [7] for a proof of this. The second inequality of Theorem 4 is a consequence of \( \tilde{\omega}(f; \delta) \leq 2 \cdot \omega(f; \delta) \). \( \square \)

In order to compare the quality of our estimate with that of Sikkema and van der Meer we consider the classical Bernstein operators as an example.

Example 1 For the Bernstein operators \( B_n \) there holds

\[
c_{n,2}(x, \delta) = \delta^2 \cdot B_n \left( s_{2,1/2} \left( \frac{e_1 - x}{\delta} \right); x \right) \leq \frac{1}{2} \frac{x(1-x)}{n} \left\{ 1 + \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\}.
\]
Proof. First recall that
\[ s_{2,2}(u) = \frac{1}{2}u^2 + \frac{1}{6} \cdot \{b_3(|u|) - b_3(|u| - [u])\}. \]

We put \( t = |u| \geq 0 \) and claim that
\[ b_3(t) - b_3(t - [t]) = 3t^2[t] - 3t[t]^2 + [t]^3 - 3t[t] + \frac{3}{2}[t]^2 + \frac{1}{2}[t] \leq t^2[t]. \]

Clearly this is true of \( 0 \leq t < 1 \). So let \( t \geq 1 \).

We divide the two sides of the inequality by \([t] \geq 1\) and multiply by 2.

Then the above inequality is equivalent to
\[ 6t^2 - 6t[t] + 2[t]^2 - 6t + 3[t] + 1 \leq 2t^2, \]
or
\[ 4t^2 - 6t + 1 \leq 6t[t] - 2[t]^2 - 3[t]. \]

Now choose \( k \in \mathbb{N} \) such that \( k \leq t < k + 1 \), then \([t] = k\), and the above reads
\[ 4t^2 - 6t + 1 \leq 6kt - 2k^2 - 3k. \]

It remains to check if this is true for all \( t \in [k, k+1)\).

For \( t = k \) we get
\[ 4k^2 - 6k + 1 \leq 6k^2 - 2k^2 - 3k, \]
which is equivalent to \( 1 \leq 3k \) (fulfilled).

For \( t = k + 1 \) we have to show that
\[ 4(k+1)^2 - 6(k+1) + 1 \leq 6k(k+1) - 2k^2 - 3k, \]
being equivalent to \(-1 \leq k \) (fulfilled).
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Hence the parabola \(4t^2 - 6t + 1\) lies below the straight line \(6kt - 2k^2 - 3k\) for \(t \in [k, k + 1]\) which is what we claimed above.

This implies that

\[
\begin{align*}
    s_{2, \frac{1}{2}}(u) & \leq \frac{1}{2}u^2 + \frac{1}{6}u^2|u| \\
    & \leq \frac{1}{2}u^2 + \frac{1}{6}|u|^3.
\end{align*}
\]

Hence

\[
c_{n,2}(x, \delta) \leq \delta^2 \cdot B_n \left( \frac{1}{2} \cdot \frac{(e_1 - x)^2}{\delta^2} + \frac{1}{6\delta^3} \cdot |e_1 - x|^3; x \right)
\]

\[
= \frac{1}{2} \left\{ \frac{x(1-x)}{n} + \frac{1}{3\delta} \cdot B_n(|e_1 - x|^3; x) \right\}
\]

Using the inequality (see [4])

\[
B_n(|e_1 - x|^3; x) \leq 3 \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \cdot B_n((e_1 - x)^2; x)
\]

we obtain

\[
c_{n,2}(x, \delta) \leq \frac{1}{2} \cdot \frac{x(1-x)}{n} \left\{ 1 + \frac{1}{3\delta} \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\}.
\]

\(\square\)
Example 2. Choose $\delta = \sqrt{\frac{2}{n}}$. Then the theorem of Sikkema and van der Meer implies

$$\left| B_n(f; x) - f(x) - \frac{x(1-x)}{2n} f''(x) \right|$$

$$\leq \frac{x(1-x)}{2n} \left\{ 1 + \sqrt{\frac{n}{2}} \cdot \sqrt{1 + \frac{x(1-x)}{n}} \right\} \cdot \omega \left( f''; \sqrt{\frac{2}{n}} \right)$$

$$\leq \left\{ 1 + \frac{1}{\sqrt{2}} \cdot \sqrt{1 + \frac{1}{4}} \right\} \cdot \frac{x(1-x)}{n} \cdot \omega \left( f''; \sqrt{\frac{2}{n}} \right)$$

$$\leq 0.9 \cdot \frac{x(1-x)}{n} \cdot \omega \left( f''; \sqrt{\frac{2}{n}} \right).$$

This is better than the corresponding result of Videnskiı̆ [9] published in 1985 and only for the Bernstein operators. In Videnskiı̆’s book instead of 0.9 the constant is one.

We now apply Theorem 4 and arrive at

**Corollary 2**

$$\left| B_n(f; x) - f(x) - \frac{x(1-x)}{2n} \cdot f''(x) \right|$$

$$\leq \frac{x(1-x)}{2n} \cdot \max \left\{ 1, \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\} \cdot \tilde{\omega}(f''; \delta)$$

$$\leq \frac{x(1-x)}{2n} \cdot \max \left\{ 2, \frac{2}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\} \cdot \omega(f''; \delta).$$

If the modulus $\omega(f''; \cdot)$ is concave, then the first inequality is better than what can be derived from Sikkema’s and van der Meer’s result because

$$\max \left\{ 1, \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\} \leq 1 + \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}.$$
However, in the general case
\[
\max \left\{ \frac{2}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\} \geq 1 + \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}.
\]
and equality is attained if and only if
\[
\delta = \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}.
\]
If we put \( \hat{c}_{n,2}(x, \delta) := 1 + \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \) and
\[
d_{n,2}(x, \delta) := \max \left\{ 1, \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\},
\]
then a possible outcome of this discussion is the following

**Theorem 5** For the Bernstein operators \( B_n, n \in \mathbb{N}, f \in C[0,1], x \in [0,1] \) and \( \delta > 0 \) there holds
\[
\left| B_n(f; x) - f(x) - \frac{x(1-x)}{2n} f''(x) \right| \\
\leq \frac{x(1-x)}{2n} \cdot \min \{ \hat{c}_{n,2}(x, \delta) \cdot \omega(f'', \delta); d_{n,2}(x, \delta) \cdot \tilde{\omega}(f'', \delta) \}.
\]

All previous quantitative Voronovskaya theorems for the Bernstein operators and \( f \in C^2[0,1] \) can be derived from Theorem 5.

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**References**


Heiner Gonska
University of Duisburg-Essen
Department of Mathematics
D-47048 Duisburg
Germany
e-mail: heiner.gonska@uni-due.de

Ioan Rașa
Technical University
Department of Mathematics
RO-400020 Cluj-Napoca
Romania
e-mail: Ioan.Rasa@math.utcluj.ro