Solution of a polylocal problem using Tchebychev polynomials

Eugen Drăghici, Daniel Pop

Abstract

Consider the problem:

\[ Ly(x) = r(x), \quad -1 \leq x \leq 1, \]
\[ y(a) = A, \quad y(b) = B \]
\[ -1 < a < b < 1, \quad a, b, A, B \in \mathbb{R}, \]

where

\[ Ly(x) := -\frac{d}{dx} \left( \frac{dy}{dx} \right) + q(x) \cdot y(x), \quad -1 \leq x \leq 1 \]

and

\[ q(x), r(x) \in C[-1, 1], \quad y(x) \in C^2[-1, 1]. \]

The aim of this paper is to present an approximate solution of this problem based on Tchebychev polynomials. We construct the approximation using Tchebychev-Gauss-Lobatto interpolation nodes. Also we use Maple 10 to obtain numerical results.

Dedicated to the memory of prof. Alexandru Lupas (1942-2007)

2000 Mathematical Subject Classification: 34B10
1 Introduction

The purpose of this paper is to approximate the solution of the following problem:

\[ \begin{cases} 
L_0(x) = r(x), & -1 \leq x \leq 1, \\
y(a) = A, & y(b) = B \\
-1 < a < b < 1, & a, b, A, B \in \mathbb{R}, 
\end{cases} \]  
(1)

where:

\[ L_0(x) := -\frac{d}{dx}(\frac{dy}{dx}) + q(x) \cdot y(x), \quad -1 \leq x \leq 1 \]  
(2)

and \( q(x), r(x) \in C[-1, 1], \ y(x) \in C^2[-1, 1], \) using a Collocation method.

This is not a two boundary value problem since \(-1 < a < b < 1\).

We have two initial value problem on \([-1, a]\) and \([b, 1]\), respectively, and on \([a, b]\) a classical boundary value problem, the existence and the uniqueness for (1) assure existence and uniqueness of these problems.

**Historical note.** In 1966, two researchers from Tiberiu Popoviciu Institute of Romanian Academy, Cluj-Napoca, Dumitru Ripianu and Oleg Arama published a paper on a polylocal problem, see ([6]).

2 Principles of the method

The implementation is inspired from ([2]). Our method is based on first kind Tchebychev polynomials ([3]) and ([5]).
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Definition 1 The polynomials $T_n(x), n \in \mathbb{N}$ defined by:

\begin{equation}
T_n(x) := \cos(n \arccos(x)), x \in [-1, 1]
\end{equation}

are called the Tchebychev polynomials of the first kind.

Definition 2 The polynomials $U_n(x), n \in \mathbb{N}$ defined by:

\begin{equation}
U_n(x) = \frac{\sin(n + 1) \cdot \arccos x}{(n + 1) \cdot \sqrt{1 - x^2}}
\end{equation}

are called the Tchebychev polynomials of the second kind.

To describe the basic method in this and later section we choose a nonuniform mesh of the given interval $[-1, 1]$ therefore:

\begin{equation}
\Delta : x_j = \cos \frac{j \cdot \pi}{n}, \; j = 1, \ldots, n.
\end{equation}

The are the zeros of Tchebychev polynomials of second kind.

The form of solution is:

\begin{equation}
u(x) = \sum_{k=0}^{n+1} c_k \cdot T_k(x)
\end{equation}

where $T_k(x)$ is the $k$-th degree first kind Tchebychev polynomials on interval $[-1, 1]$.

We shall choose the basis such that the following conditions hold:

- the solution verifies the differential equation

\begin{equation}
Lu(x_j) = r_j, j = 1, 2, \ldots n
\end{equation}
the solution verifies

\begin{equation}
\begin{aligned}
u(a) &= A, \\
u(b) &= B.
\end{aligned}
\end{equation}

We choose this mesh (3), because in ([2], pag30) prove that interpolation at Tchebychev points is nearly optimal. Since the mesh (3) has \( n \) points, we include the points \( a, b \) and suppose that \( a, b \neq x_j \) for all \( j = 1, 2, \ldots n \).

**Remark 1**

- If \( a = x_j, b = x_j \) then we increment \( n \).
- The method do not depend on conditions on \( q(x) \).
- The Tchebychev polynomials are generated via the orthopoly package with the Maple sequence:

\begin{verbatim}
> S := (x, k, a, b) -> T(k, ((b-a)*x + a + b)/2):;
\end{verbatim}

**3 Numerical Results**

We shall give two examples. For each example we plot the exact and approximate solution and generate the execution profile with the pair `profile-showprofile` see ([4])

- **First we approximate a oscillating solution**:

\begin{equation}
\begin{aligned}
-Z''(t) - 243 \cdot Z(t) &= t; -1 \leq t \leq 1 \\
Z(-1) &= Z(1) = 0
\end{aligned}
\end{equation}
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with conditions:

\[ Z \left(-\frac{1}{4}\right) = -\sin\left(\frac{\sqrt{3}}{4}\right) + \frac{1}{243 \sin(9\sqrt{3})} + \frac{1}{972} \quad (10) \]

\[ Z \left(\frac{1}{2}\right) = \frac{\sin\left(\frac{\sqrt{3}}{2}\right)}{243 \sin(9\sqrt{3})} - \frac{1}{486} \quad (11) \]

The exact solution provided by \textit{dsolve} is:

\[ Z(t) = \frac{\sin(9\sqrt{3}t) - t \sin(9\sqrt{3})}{243 \sin(9\sqrt{3})} \]

Since

\[ \int_{-1}^{1} |q(x)| \, dx > 2 \]

using disconjugate criteria given by Lyapunov (1893) the problem (9) has an oscillatory solution. We used Maple 8 to solve the problem exactly and to approximate the solution, for \( n = 17 \) and \( n = 50 \). We also plot the error in semilogarithmic scale.

The code Maple is:

\[
\text{restart; with(orthopoly); with(CodeTools); with(plots):}
\]
\[
S := (x, k, a, b) \rightarrow T(k, ((b-a)*x+a+b)/2);
\]
\[
S := (x, k, a, b) \rightarrow T(k, 1/2 \cdot (b - a) \cdot x + 1/2 \cdot a + 1/2 \cdot b)
\]
\[
genceb := \text{proc}(x, n, q, r, c0, d0, alpha, beta)
\]
\[
\text{local k, ecY, ecd, C, h, Y, c, a, b;}
\]
\[
\text{global S;}
\]
\[
a := x[0]; b := x[n-1];
\]
\[
Y := 0;
\]
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> for k from 0 to n+1 do
> Y:=Y+c[k]*S(t,k,a,b);
> end do;
> Y:=simplify(Y);
> ecY:=-diff(Y,t$2)+q(t)*Y=r(t):
> ecd:=Array(0..n+1);
> for k from 0 to n-1 do
> ecd[k]:=eval(ecY,t=x[k]):
> end do;
> ecd[n]:=eval(Y,t=c0)=alpha:
> ecd[n+1]:=eval(Y,t=d0)=beta:
> C:=solve({seq(ecd[k],k=0..n+1)},{seq(c[k],k=0..n+1)});
> assign(C):
> return Y:
> end proc:

#we define the function from differential equations
> q:=t->-243; r:=t->t:
> ecz:=-diff(Z(t),t$2)+q(t)*Z(t)=r(t):
> dsolve({ecz,Z(-1)=0,Z(1)=0},Z(t)):simplify(%):
> assign(%):
> n:=17: b:=1: a:=-1: c0:=-1/4: d0:=1/2:alpha:=eval(Z(t),t=c0):
> beta=eval(Z(t),t=d0);
> u:=[-1/4,seq((b-a)/2*cos(k*Pi/n)+(a+b)/2,k=1..n),1/2];
> u:=sort(evalf(u)):
> n:=nops(u); x:=Array(0..n-1,u):
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> eval(x):
> profile(genceb): Y:=genceb(x,n,q,r,-1/4,1/2,alpha,beta)
> plot(Y,t=-1..1,title=" Approx TCHEBYCHEV");
> plot(Z(t),t=-1..1,color=[GREEN],title="Exact Solution");
> p1:=plot([Y,Z(t)],t=-1..1,title="Exact&Approx.solution;n=17");
> p2:=plots[pointplot]([[-1/4,eval(Z(t),t=-1/4)],[1/2,eval(Z(t),t=1/2)]],symbol=circle,symbolsize=30,color=[BLACK]);
> plots[display]({p1,p2});showprofile(genceb);
> plot(log(Y-Z(t)),t=-0.99..0.99,title="Error in Semilog. scale;n=17",color=[BLUE]);
> #quit

Fig. 1. The graph of exact and approximate solution, oscillating problem, n=17
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Fig. 2. Error plot, oscillating problem, n=17

Here are the profiles for the procedure in the case of oscillating solution:

<table>
<thead>
<tr>
<th>function</th>
<th>depth</th>
<th>calls</th>
<th>time</th>
<th>time%</th>
<th>bytes</th>
<th>bytes%</th>
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<td>genceb</td>
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<td>total</td>
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<td>100.00</td>
</tr>
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</table>

For $n = 50$, we obtain a very good approximation, but we must increase the number of decimals with Maple command:

$$> \text{Digits} := 18;$$
Fig. 3. The graph of exact and approximate solution, oscillating problem, n=50

Fig. 4. Error plot, oscillating problem, n=50
Here are the profiles for the procedures in the case of oscillating problem.

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<table>
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<tr>
<th>function</th>
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<td>28.749</td>
<td>100.00</td>
<td>455186448</td>
<td>100.00</td>
</tr>
</tbody>
</table>

- The second example is a nonoscillating solution.

Example is from ([1, page 560]).

\[
-\ddot{y} - y = x, \quad x \in [-1, 1] \\
y(-1) = y(1) = 0
\]

(12)

with conditions:

\[
y\left(\frac{-1}{4}\right) = -\frac{\sin \frac{1}{4}}{\sin 1} + \frac{1}{4} \\
y\left(\frac{1}{2}\right) = \frac{\sin \frac{1}{2}}{\sin 1} - \frac{1}{2}
\]

The exact solution given by \texttt{dsolve} is \( y(t) = -\frac{\sin(t) + t \sin 1}{\sin 1} \). Since

\[
\int_{-1}^{1} |q(x)| \, dx \leq 2
\]

using disconjugate criteria given by Lyapunov (1893) the problem (12) has an nonoscillatory solution. For \( n = 10 \), we plot the graph of exact solution and approximation.
Fig. 5. The graph of exact and approximate solution, non oscillating problem, n=10

Fig. 6. Error plot, non oscillating problem
The profile in this case is:

<table>
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</tr>
</tbody>
</table>

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References


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