About Fejér’s sum

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Abstract

In this paper we will show an improvement of Fejér inequality,
\[
\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{16\pi(1 - P_n(\cos \phi))}{(n + 1)\sin \phi} \quad \text{or}
\]
\[
\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{\pi(1 - \cos n\phi)}{2^{2n-5}(n + 1)\sin \phi}, \quad \text{where}
\]
\[P_k(x) = \frac{1}{2^k k!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n\] is Legendre polynomial

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1 Main results

Using the quadrature formulas of Bouzitat, we present an improvement of Fejér inequality:
\[
\sum_{k=1}^{n} \frac{\sin k\phi}{k} > 0, \quad \forall \phi \in (0, \pi), \quad n \in \mathbb{N}^*
\]
For \((\alpha, \beta) \in (-1, +\infty) \times (-1, +\infty)\) we define the functional \(I^{(\alpha,\beta)} : \Pi \rightarrow \mathbb{R}\). It is known that

\[
I^{(\alpha,\beta)}(f) = \int_{-1}^{1} f(x)(1-x)^\alpha(1+x)^\beta \, dx, \ f \in \Pi_n
\]

where \(\Pi_n\) is the set of all polynomials of degree less or equal to \(n\). The following result is well-known.

**Theorem 1.** Let \(P \in \Pi_n\) with degree \([P] = n\) and \(P \geq 0\) on \([-1, 1]\). If \(m = \left\lfloor \frac{n}{2} \right\rfloor,\ d = \left\lfloor \frac{n+1}{2} \right\rfloor\) then

\[
I^{(\alpha,\beta)}(P) \geq 2^{\alpha+\beta+1} B(\alpha + 1, \beta + 1) \frac{\Gamma(m + 1)(\beta + 1)d}{(\alpha + 2)m(\alpha + \beta + 2)d}
\]

We note with \(P_k\) the Legendre polynomial, and with \(U_k\) the Cebyshev second species polynomial. We have

\[
P_k(x) = \frac{1}{2^k k!} \left(\frac{d}{dx}\right)^{n} (x^2 - 1)^n,
\]

\[
U_k(x) = \frac{\sin((k + 1) \arccos x)}{(k + 1) \sqrt{1 - x^2}}.
\]

It is well-known (see [9]) that \(|P_n(x)| \leq 1,\ x \in (-1, 1)\) and \(|P_n(\pm)| = 1\).

**Theorem 2.** For all \(x \in (-1, 1)\) and \(n \in \mathbb{N}\) the inequality

\[
\sum_{k=0}^{n} U_k(x) \geq \frac{16\pi}{n+2} \cdot \frac{1 - P_{n+1}(x)}{1 - x^2} \text{ holds.}
\]

**Proof.** For \(x \in (-1, 1)\). A. Lupaş [6] established the identity

\[
\sum_{k=1}^{n} \frac{\sin(k \arccos x)}{k} = \frac{\sqrt{1 - x}}{2} \int_{-1}^{1} \frac{1 - P_n(y)}{1 - y} \cdot \frac{dy}{\sqrt{x - y}}.
\]
For \( y(t, x) := \frac{x+1}{2} t + \frac{x-1}{2} \), we have

\[
\sum_{k=0}^{n} U_k(x) = \frac{\sqrt{2}}{2(1+x)} \int_{-1}^{1} H_n(t, x) \frac{dt}{\sqrt{1-t}}, \quad H_n(t, x) := \frac{1 - P_{n+1}(y(t, x))}{1 - y(t, x)}.
\]

We observe that the \( H_n(\cdot, x) \) is a polynomial of effective degree \( n \), and, additionally \( H_n(\cdot, x) \geq 0 \) on \([-1, 1]\). But \( H_n(1, x) = \frac{1 - P_{n+1}(x)}{1 - x} \). We have

\[
\sum_{k=0}^{n} U_k(x) \geq \mu_n \cdot \frac{1 - P_{n+1}(x)}{1 - x^2}
\]

where denoting \( m = \lfloor n/2 \rfloor \), \( d = \lfloor (n+1)/2 \rfloor = n - m \) we have

\[
\mu_n := \frac{2^{2n+3}m!(m+1)!(d+1)!d!}{(2m+2)!(2d+2)!} = \frac{8\pi \Gamma(m+1)\Gamma(d+1)}{\Gamma \left( m + \frac{3}{2} \right) \Gamma \left( d + \frac{3}{2} \right)}.
\]

From the convexity of \( \log \Gamma : (0, \infty) \to (0, \infty) \) we have the next evaluations

\[
\frac{1}{\sqrt{x + \frac{1}{2}}} \leq \frac{\Gamma \left( x + \frac{1}{2} \right)}{\Gamma(x+1)} \leq \frac{1}{\sqrt{x}}.
\]

which implies:

\[
\frac{8\pi}{\sqrt{(m+1)(d+1)}} \leq \mu_n \leq \frac{8\pi}{\sqrt{(m+1/2)(d+1/2)}}, \quad \mu_n \geq \frac{16\pi}{n+2},
\]

So \( \sum_{k=0}^{n} U_k(x) \geq \frac{16\pi}{n+2} \cdot \frac{1 - P_{n+1}(x)}{1 - x^2} \) and thus, the assertion is proved.

**Corollary 1.** For all \( \phi \in [0, \pi] \) the following inequality is true:

\[
\sum_{k=1}^{n} \frac{\sin k\phi}{k} \geq \frac{16\pi(1 - P_n(\cos \phi))}{(n+1) \sin \phi}.
\]
For proving our main result we will need the following lemmas:

**Lemma 1.** (T. Koorwinder [1]). Let $g_{k,n}^{(\alpha,\beta)}$ be the coefficients of the following development:

$$R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} g_{k,n}^{(\alpha,\beta)} R_k^{(a,b)}(x).$$

If $a \leq b$, $\alpha + \beta \geq a + b$ and $\beta - \alpha \leq b - a$, then $g_{n,k}^{(\alpha,\beta)} \geq 0$, $0 \leq k \leq n$.

**Lemma 2.** If $\alpha \geq a \geq -\frac{1}{2}$, and

$$T_n^{(\alpha)}(x) := \frac{(\alpha + 1)_n}{(n + 2\alpha + 1)_n} (1 - R_n^{(\alpha,\alpha)}(x)),$$

then $T_n^{(\alpha)}(x) \geq T_n^{(\alpha)}(x)$ for all $x \in [-1, 1]$.

**Proof.** Taking into account the hypergeometric form of the Jacobi polynomials, through identification of $x^n$ coefficients we deduce:

$$g_{n,n}^{(\alpha,\beta)} = \frac{(a+1)_n(n+\alpha+\beta+1)_n}{(\alpha+1)_n(n+a+b+1)_n}.$$  \hspace{1cm} (2)

On the other hand, for $x = 1$ and taking into account that $R_k^{(a,b)}(1) = 1$ we deduce that $\sum_{k=0}^{n} g_{k,n}^{(\alpha,\beta)} = 1$. We consider $a = b \geq -\frac{1}{2}$ and $\alpha = \beta$. For $\alpha \geq a \geq -\frac{1}{2}$ and $x \in [-1, 1]$ we have

$$1 - R_n^{(\alpha,\alpha)}(x) = \sum_{k=0}^{n} g_{k,n}^{(\alpha,\alpha)} (1 - R_k^{(\alpha,a)}(x)) \geq g_{n,n}^{(\alpha,\alpha)} (1 - R_n^{(\alpha,a)}(x))$$

and $T_n^{(\alpha)}(x) \geq T_n^{(\alpha)}(x)$, $\forall x \in [-1, 1]$. So, if we choose $\alpha = 0$ and $a \in [-1/2, 0]$ for $x \in [-1, 1]$ we have

$$1 - P_n(x) \geq \frac{(a+1)_n(n+1)_n}{(n+2a+1)_nn!} (1 - R_n^{(\alpha,a)}(x))$$
and 

$$1 - P_n(x) \geq \frac{1}{2^{2n-1}}(1 - T_n(x)).$$

**Corollary 2.** For all $\phi \in [0, \pi]$ the next inequality is true

$$\sum_{k=1}^{n} \sin k\phi \frac{\sin k\phi}{k} \geq \frac{\pi(1 - \cos n\phi)}{2^{2n-5}(n + 1)\sin \phi}$$

**References**


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