A note on Mathieu’s inequality

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Abstract

In this note we obtain a generalization for Mathieu’s inequality.

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1. Introduction

Mathieu[16] conjectured in 1890 that the inequality

\[ \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} < \frac{1}{c^2} \]  

where \( c \) is a real number, \( c \neq 0 \), is valid. It was proved only in 1952 by L. Berg[3]. E. Makai [15] gave a very elegant and elementary proof for (1) and obtained the following lower estimation:

\[ \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} > \frac{1}{c^2 + \frac{1}{2}} \]

P. H. Dianada [7] refined Mathieu’s inequality (1) to

\[ \sum_{n=1}^{\infty} \frac{2n}{(n^2 + c^2)^2} > \frac{1}{c^2} - \frac{1}{(2c^2 + 2c + 1)(8c^2 + 8c + 3)} \]
Gh. Costovici [6] has proved the following inequalities of Mathieu type:

\[
\sum_{n=1}^{\infty} \frac{4n(n+1)(n+2)}{(n^2+2n+c^2)^4} < \frac{1}{c^4}
\]

and

\[
\sum_{n=1}^{\infty} \frac{6n(n+1)(n+2)(n+3)(n+4)}{(n^2+5n+c^2)^6} < \frac{1}{c^6}.
\]

H. Alzer, J.L. Brenner and O. G. Ruchr [2] showed that the best constants \(a\) and \(b\) in

\[
\frac{1}{c^2 + a} < \sum_{n=1}^{\infty} \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2 + b}, \quad c \neq 0
\]

are \(a = \frac{1}{2}\xi(3)\) and \(b = \frac{1}{6}\), where \(\xi(\cdot)\) denotes the Riemann Zeta function defined by

\[
\xi(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.
\]

Using the integral expression for Mathieu's series, many authors obtained interesting refinements and extensions of Mathieu's inequality ([1], [12], [13], [14], [18], [19]).

In [1] D. Acu proved the inequality

\[
\sum_{n=1}^{\infty} \frac{(p+1)n_{[p]}}{(n_{[p]}(n + p - 1) + c^2)^2} < \frac{1}{c^2}
\]

\(c \neq 0, p \geq 1\), where \(n_{[p]} = n(n+1)\ldots(n+p-1)\).

P. Dicu and M. Acu in [8] obtained

\[
\frac{1}{a_1^2 + c^2 + \frac{c^2}{2} - a_1 r} < \sum_{n=1}^{\infty} \frac{2a_n r}{(a_n^2 + c^2)^2} < \frac{1}{a_1^2 + c^2 - a_1 r},
\]
\(c \neq 0\), where \((a_n)_{n \geq 1}\) is an arithmetic progression with \(a_1 > 0\) and the ration \(r > 0\).

In this note, we present the proofs more simple for the inequalities (2) and (3), and give new inequalities of type (2)-(3).

2. A simple proof of the inequality (2)

We have

\[(n^2 + 2n + c^2)^4 > (n^2 + 2n)^4 + 6n^2(n + 2)^2 \cdot c^4 + c^8\]

and

\[(n^2 + 2n)^4 = n^4(n+2)^4 = n^2 \cdot n^2(n+2)^2(n+2)^2 > n^2(n^2-1)(n+2)^2(n^2+4n+3) =
(n - 1)n(n + 1)(n + 2)n(n + 1)(n + 2)(n + 3).

But \(6n^2(n + 2)^2 > (n - 1)n(n + 1)(n + 2) + n(n + 1)(n + 2)(n + 3)\) because it is equivalent to

\[4n^2 + 8n - 2 > 0,\] which is true for \(n \geq 1\).

Now, it results

\[(4) \quad (n^2 + 2n + c^2)^4 > [(n - 1)n(n + 1)(n + 2) + c^4][n(n + 1)(n + 2)(n + 3) + c^4].\]

Form (4), it follows that

\[
\sum_{n=1}^{\infty} 4n(n+1)(n+2) < \sum_{n=1}^{\infty} \frac{4n(n+1)(n+2)}{[(n - 1)n(n + 1)(n + 2) + c^4][n(n + 1)(n + 2)(n + 3) + c^4]}
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{(n - 1)n(n + 1)(n + 2) + c^4} - \frac{1}{n(n + 1)(n + 2)(n + 3) + c^4} \right) < \frac{1}{c^4}, \text{ q.e.d.}
\]

3. A simple proof of (3)
Observe that

\[ (n^2 + 5n + c^2)^6 > (n^2 + 5n)^6 + 20(n^2 + 5n)^3 \cdot c^6 + c^{12}. \]

Since \( n^2 > n^2 - 1 \) and \( n^2(n + 5)^5 > (n + 1)(n + 2)^2(n + 3)^2(n + 4)^2 \) which is equivalent to

\[ 6n^6 + 99n^5 + 631n^4 + 1771n^3 + 1489n^2 - 1344n - 576 > 0 \]

for \( n \in \mathbb{N}, n \geq 1 \), we obtain

\[ (n^2 + 5n)^6 = n^2n^2(n^2 + 5)^5(n + 5) > n^2(n^2 - 1)(n + 1)(n + 2)^2. \]

\[ \cdot(n + 3)^2(n + 4)^5(n + 5) = [(n - 1)n(n + 1)(n + 2)(n + 3)(n + 4)][n(n + 1)(n + 2)(n + 3)(n + 4)(n + 5)]. \]

Now, we deduce

\[ 20n^3(n + 5) > (n - 1)n(n + 1)(n + 2)(n + 3)(n + 4) \]

\[ +n(n + 1)(n + 2)(n + 3)(n + 4)(n + 5), \] because it is equivalent to

\[ 9n^5 + 136n^4 + 695n^3 + 1130n^2 - 124n - 48 > 0 \]

which is true for \( n \in \mathbb{N}, n \geq 1 \).

From (5), (6) and (7) we obtain the inequality

\[ (n^2 + 5n + c^2)^6 > [(n - 1)n(n + 1)(n + 2)(n + 3)(n + 4) + c^6] \]

\[ \cdot[n(n + 1)(n + 2)(n + 3)(n + 4)(n + 5) + c^6]. \]

Using (8) we have

\[ \sum_{n=1}^{\infty} \frac{6n(n + 1)(n + 2)(n + 3)(n + 4)}{(n^2 + 5n + c^2)^6} < \]
\[\sum_{n=1}^{\infty} \frac{6n(n+1)(n+2)(n+3)(n+4)}{[(n-1)n(n+1)(n+2)(n+3)(n+4)+c^6][n(n+1)(n+2)(n+3)(n+4)(n+5)+c^6]} = \]
\[\sum_{n=1}^{\infty} \left( \frac{1}{(n-1)n(n+1)(n+2)(n+3)(n+4)+c^6} - \frac{1}{n(n+1)(n+2)(n+3)(n+4)(n+5)+c^6} \right) < \frac{1}{c^6} \]
and the inequality (3) is proved.

4. A new inequality by type (2) and (3)

By a reasoning similar to the proof of (2) and (3) we also can prove the following inequality:
\[\sum_{n=1}^{\infty} \frac{8n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(n^2 + 7n + c^2)^8} < \frac{1}{c^8}.\]

For the proof we observe the following inequalities
\[(n^2 + 7n + c^2)^8 > (n^2 + 7)^8 + 70(n^2 + 7n)^4c^8 + c^{10},\]
\[(n^2 + 7n)^8 > (n-1)n^2(n+1)^2(n+2)^2(n+3)^2(n+4)^2(n+5)^2(n+6)^2(n+7)\]
and
\[70(n^2 + 7n)^4 > (n-1)n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)\]
\[+n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)\]
are valid for \(n \in \mathbb{N}, n \geq 1\).

5. The open problem

We denote
\[n_{[p]} = n(n+1) \ldots (n+p-1), p \in \mathbb{N}^*, n \in \mathbb{N}^*.\]
Is the inequality
\[
\sum_{n=1}^{\infty} \frac{(2p+2)n_{[2p+1]}}{(n^2 + (2p + 1)n + c^2)^{2p+2}} < \frac{1}{c^{2p+2}}, \quad c \neq 0
\] true?

6. The other elementary inequalities of Mathieu’s type

6.1 If \( c \neq 0 \), then we have
\[
\sum_{n=1}^{\infty} \frac{n}{(3 \cdot 5 \cdot 7 \ldots (2n+1) + c^2)^2} < \frac{1}{2(c^2 + 1)}.
\]

Proof of (9) follows from

\[
\frac{n}{[3 \cdot 5 \cdot 7 \ldots (2n-1) + c^2]^2} < \frac{n}{[3 \cdot 5 \cdot 7 \ldots (2n+1) + c^2][3 \cdot 5 \cdot 7 \ldots (2n+1) + c^2]} =
\]

\[
= \frac{1}{2} \left( \frac{1}{3 \cdot 5 \cdot 7 \ldots (2n-1) + c^2} - \frac{1}{3 \cdot 5 \cdot 7 \ldots (2n+1) + c^2} \right).
\]

6.2 If \( c \neq 0 \) and \( a > 0 \), then we have
\[
\sum_{n=1}^{\infty} \frac{2an}{(an^2 + an + c^2)^2} < \frac{1}{c^2}.
\]

Proof of (10) follows from

\[
\frac{2an}{(an^2 + an + c^2)^2} < \frac{2an}{(an^2 - an + c^2)(an^2 + an + c^2)} = \frac{1}{an^2 - an + c^2} \cdot \frac{1}{an^2 + an + c^2} =
\]

\[
= \frac{1}{an^2 - an + c^2} - \frac{1}{a(n+1)^2 - a(n+1) + c^2}.
\]
References


[10] O. E. Emersleban, Über die Reihe \( \sum_{k=1}^{\infty} \frac{k}{(k^2 + c^2)^2} \), Math. Ann. 125(1952), 165-171.


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