On the Fekete-Szegö inequality for a class of analytic functions defined by using the generalized Sălăgean operator

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Abstract

In this paper we obtain the Fekete-Szegö inequality for a class of analytic functions $f(z)$ defined in the open unit disk for which

$$\left(\frac{D_{\lambda}^{n+1}f}{D_{\lambda}^{n}f}\right)^{\alpha} \left(\frac{D_{\lambda}^{n+2}f}{D_{\lambda}^{n+1}f}\right)^{\beta} \quad (\alpha, \beta, \lambda \geq 0)$$

lies in a region starlike with respect to 1 and which is symmetric with respect to the real axis.

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1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $S$ be the subclass of $A$ consisting of univalent functions.

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The generalized Sălăgean differential operator is defined in [2] by
\[ D_0^\lambda f(z) = f(z), \quad D_1^\lambda f(z) = (1 - \lambda) f(z) + \lambda z f'(z) \]
\[ D_n^\lambda f(z) = D_1^\lambda(D_{n-1}^\lambda f(z)), \quad \lambda \geq 0. \]

If \( f \) is given by (1) we see that
\[
D_n^\lambda f(z) = z + \sum_{k=2}^{\infty} [1 + (k - 1)\lambda] a_k z^k.
\]

When \( \lambda = 1 \) we get the classic Sălăgean differential operator [6].

Let \( \Phi(z) \) be an analytic function with positive real part on \( U \) with \( \Phi(0) = 1 \), \( \Phi'(0) > 0 \) which maps the unit disk \( U \) onto a region starlike with respect to 1 which is symmetric with respect to the real axis.

Denote by \( S^*(\Phi) \) the class of functions \( f \in S \) for which
\[
\frac{zf'(z)}{f(z)} \prec \Phi(z), \quad z \in U
\]
and denote by \( C(\Phi) \) the class of functions \( f \in S \) for which
\[
1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z), \quad z \in U
\]
where " \( \prec \) " stands for the usual subordination. The classes \( S^*(\Phi) \) and \( C(\Phi) \) were defined and studied by Ma and Minda [1]. They obtained the Fekete-Szegö inequality for functions in the class \( S^*(\Phi) \) and also for functions in the class \( C(\Phi) \).

By using the generalized Sălăgean differential operator we define the following class of functions:

**Definition 1.1.** Let \( \Phi(z) \) be a univalent starlike function with respect to 1 which maps the unit disk onto a region in the right half plane symmetric with respect to the real axis, \( \Phi(0) = 1 \) and \( \Phi'(0) > 0 \). A function \( f \in A \) is in the class \( M_{\alpha,\beta}^{n,\lambda}(\Phi) \) if
\[
\left( \frac{D_{n+1}^\lambda f(z)}{D_n^\lambda f(z)} \right)^\alpha \left( \frac{D_{n+2}^\lambda f(z)}{D_{n+1}^\lambda f(z)} \right)^\beta \prec \Phi(z),
\]

\( 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, \lambda > 0. \)
It follows that
\[ M_{0,1}^{0,1}(\Phi) \equiv C(\Phi) \quad \text{and} \quad M_{1,0}^{0,1}(\Phi) \equiv S^*(\Phi). \]

When \( n = 0 \) and \( \lambda = 1 \) we obtain the class \( M_{\alpha,\beta}(\Phi) \) studied by Ravichadran et.al. [3].

In this paper we obtain the Fekete-Szegő inequality for functions in the class \( M_{n,\lambda}^{\alpha,\beta}(\Phi) \).

To prove our results we shall need the following lemmas.

**Lemma 1.1.** [1] If \( p_1(z) = 1 + c_1z + c_2z^2 + \ldots \) is an analytic function with positive real part in \( U \), then
\[
|c_2 - vc_1^2| \leq \begin{cases} 
-4v + 2, & \text{if } v \leq 0 \\
2, & \text{if } 0 \leq v \leq 1 \\
4v - 2, & \text{if } v \geq 1.
\end{cases}
\]

When \( v < 0 \) or \( v > 1 \), the equality holds if and only if \( p_1(z) = (1+z)/(1-z) \) or one of its rotations. If \( 0 < v < 1 \), then the equality holds if and only if \( p_1(z) = (1+z^2)/(1-z^2) \) or one of its rotations. If \( v = 0 \), the equality holds if and only if
\[
p_1(z) = \left( \frac{1+a}{2} \right) \frac{1+z}{1-z} + \left( \frac{1-a}{2} \right) \frac{1-z}{1+z}, \ 0 \leq a \leq 1
\]
or one of its rotations. If \( v = 1 \), the equality holds if and only if \( p_1 \) is the reciprocal of one of the functions such that the equality holds in the case of \( v = 0 \).

Also the above upper bound is sharp and it can be improved as follows when \( 0 < v < 1 \):
\[
|c_2 - vc_1^2| + v|c_1^2| \leq 2, \ 0 < v \leq \frac{1}{2}
\]
and
\[
|c_2 - vc_1^2| + (1-v)|c_1^2| \leq 2, \ \frac{1}{2} < v \leq 1.
\]
Lemma 1.2. [4] If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is an analytic function with positive real part in \( U \), then
\[
|c_2 - vc_1^2| \leq 2 \max \{1; |2v - 1|\}.
\]
The result is sharp for the function
\[
p_1(z) = \frac{1 + z^2}{1 - z^2} \text{ or } p_1(z) = \frac{1 + z}{1 - z}.
\]

2 Fekete-Szegő problem

We prove our main result by making use of Lemma 1.1.

Theorem 2.1. Let \( \Phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) If \( f(z) \) given by (1.1) is in the class \( M_{n, \beta}^{n, \alpha}(\Phi) \), then
\[
|a_3 - \mu a_2^2| \leq
\]
\[
\leq \begin{cases}
\frac{1}{4\lambda(1+2\lambda)^{n[\alpha+\beta(1+2\lambda)]}} \left[ 2B_2 - \frac{B_1^2}{\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2} \gamma} \right], & \text{if } \mu \leq \sigma_1 \\
\frac{1}{2\lambda(1+2\lambda)^{n[\alpha+\beta(1+2\lambda)]}} \left[ -2B_2 + \frac{B_1^2}{\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2} \gamma} \right], & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\
\frac{1}{4\lambda(1+2\lambda)^{n[\alpha+\beta(1+2\lambda)]}} \left[ 1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2}} \right] a_2 \right] |a_2|^2
\end{cases}
\]

Further, if \( \sigma_1 < \mu \leq \sigma_3 \), then
\[
|a_3 - \mu a_2^2| + \\
+ \frac{\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2}}{2(1+2\lambda)^{n[\alpha+\beta(1+2\lambda)]B_1}} \left[ 1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2}} \right] |a_2|^2
\]

If \( \sigma_3 < \mu \leq \sigma_2 \), then
\[
|a_3 - \mu a_2^2| + \\
+ \frac{\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2}}{2(1+2\lambda)^{n[\alpha+\beta(1+2\lambda)]B_1}} \left[ 1 + \frac{B_2}{B_1} - \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n[\alpha+\beta(1+\lambda)]^2}} \right] |a_2|^2
\]
On the Fekete-Szegö inequality...

\[ \leq \frac{B_1}{2\lambda(1 + 2\lambda)^{2n}[\alpha + \beta(1 + 2\lambda)]}, \]

where

\[ \sigma_1 := \frac{2\lambda(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)]^2(B_2 - B_1)}{4(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]B_1^2} \]
\[ - \frac{B_1^2(1 + \lambda)^{2n}[\lambda[\alpha + \beta(1 + \lambda)]^2 - (\lambda + 2)[\alpha + \beta(1 + \lambda)^2]]}{4(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]B_1^2} \]
\[ \sigma_2 := \frac{2\lambda(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)]^2(B_2 + B_1)}{4(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]B_1^2} \]
\[ - \frac{B_1^2(1 + \lambda)^{2n}[\lambda[\alpha + \beta(1 + \lambda)]^2 - (\lambda + 2)[\alpha + \beta(1 + \lambda)^2]]}{4(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]B_1^2} \]
\[ \sigma_3 := \frac{2\lambda(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)]B_2}{4(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]B_1^2} \]
\[ - \frac{B_1^2(1 + \lambda)^{2n}[\lambda[\alpha + \beta(1 + \lambda)]^2 - (\lambda + 2)[\alpha + \beta(1 + \lambda)^2]]}{4(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]B_1^2} \]

and

\[ \gamma := \lambda(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)^2] - (\lambda + 2)(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)^2] + 4\mu(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]. \]

These results are sharp.

**Proof.** Let \( f \in M_{\alpha, \beta}^{n, \lambda}(\Phi) \) and let

\[ p(z) := \left( \frac{D_{\lambda}^{n+1}f(z)}{D_{\lambda}^nf(z)} \right)^\alpha \left( \frac{D_{\lambda}^{n+2}f(z)}{D_{\lambda}^{n+1}f(z)} \right)^\beta = 1 + b_1z + b_2z^2 + \ldots \]

Since the function \( \Phi(z) = 1 + B_1z + B_2z^2 + \ldots \) is univalent and \( p \prec \Phi \) then the function

\[ p_1(z) = \frac{1 + \Phi^{-1}(p(z))}{1 - \Phi^{-1}(p(z))} = 1 + c_1z + c_2z^2 \ldots \]

is analytic and has positive real part in \( U \). We also have

\[ p(z) = \Phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2}B_1c_1z + \left[ \frac{1}{2}B_1c_2 - \frac{1}{2}c_1^2 \right] + \frac{1}{4}B_2c_1^2 \right] z^2 + \ldots \]
From (2.1) we obtain
\[ b_1 = \frac{1}{2}B_1c_1 \quad \text{and} \quad b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2. \]

By making use of (1.1) and (1.2) we obtain
\[
\frac{D_{n+1}^\lambda f(z)}{D_\lambda^nf(z)} = 1 + \lambda(1 + \lambda)^n a_2 z + [2\lambda(1 + 2\lambda)^n a_3 - \lambda(1 + \lambda)^{2n} a_2^2] z^2 + \ldots
\]
and therefore we have
\[
\left(\frac{D_{n+1}^\lambda f(z)}{D_\lambda^nf(z)}\right)^\alpha = 1 + \alpha \lambda(1 + \lambda)^n a_2 z + \ldots
\]
Similarly we obtain
\[
\left(\frac{D_{n+2}^\lambda f(z)}{D_{n+1}^\lambda f(z)}\right)^\beta = 1 + \beta \lambda(1 + \lambda)^{n+1} a_2 z + \ldots
\]
Thus we have
\[
\left(\frac{D_{n+1}^\lambda f(z)}{D_\lambda^nf(z)}\right)^\alpha \left(\frac{D_{n+2}^\lambda f(z)}{D_{n+1}^\lambda f(z)}\right)^\beta = 1 + \lambda(1 + \lambda)^n [\alpha + \beta(1 + \lambda)] a_2 z + \ldots
\]
In view of (2.1) it results
\[
24 \quad (2.2) \quad b_1 = \lambda(1 + \lambda)^n [\alpha + \beta(1 + \lambda)] a_2
\]
and
\[
(2.3) \quad b_2 = 2\lambda(1 + 2\lambda)^n [\alpha + \beta(1 + 2\lambda)] a_3 + \lambda^2 [\alpha + \beta(1 + \lambda)]^2 - \lambda(\lambda + 2)[\alpha + \beta(1 + \lambda)]^2 (1 + \lambda)^{2n} a_2^2.
\]
Therefore we have

\[
(2.4) \quad a_3 - \mu a_2^2 = \frac{B_1}{4\lambda (1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]} [c_2 - vc_1^2]
\]

where

\[
v := \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)]^2} \right].
\]

Our result follows now by an application of Lemma 1.1. To show that the bounds are sharp, we consider the functions \(K_{\Phi,m}(m = 2, 3, \ldots)\) defined by

\[
\left( \frac{D_{\lambda}^{n+1} K_{\Phi,m}(z)}{D_{\lambda}^{n} K_{\Phi,m}(z)} \right)^{\alpha} \left( \frac{D_{\lambda}^{n+2} K_{\Phi,m}(z)}{D_{\lambda}^{n+1} K_{\Phi,m}(z)} \right)^{\beta} = \Phi(z^{m-1}),
\]

\[K_{\Phi,m}(0) = [K_{\Phi,m}]'(0) - 1 = 0\]

and the functions \(F_\delta, G_\delta\) \((0 \leq \delta \leq 1)\) defined by

\[
\left( \frac{D_{\lambda}^{n+1} F_\delta(z)}{D_{\lambda}^{n} F_\delta(z)} \right)^{\alpha} \left( \frac{D_{\lambda}^{n+2} F_\delta(z)}{D_{\lambda}^{n+1} F_\delta(z)} \right)^{\beta} = \Phi \left( \frac{z(z + \delta)}{1 + \delta z} \right), F_\delta(0) = F_\delta'(0) - 1 = 0
\]

and

\[
\left( \frac{D_{\lambda}^{n+1} G_\delta(z)}{D_{\lambda}^{n} G_\delta(z)} \right)^{\alpha} \left( \frac{D_{\lambda}^{n+2} G_\delta(z)}{D_{\lambda}^{n+1} G_\delta(z)} \right)^{\beta} = \Phi \left( - \frac{z(z + \delta)}{1 + \delta z} \right), G_\delta(0) = G_\delta'(0) - 1 = 0.
\]

It is clear that the functions \(K_{\Phi,m}, F_\delta\) and \(G_\delta\) belong to the class \(M_{n,\lambda}^{\alpha,\beta}(\Phi)\). If \(\mu < \sigma_1\) or \(\mu > \sigma_2\), then the equality holds if and only if \(f\) is \(K_{\Phi,2}\) or one of its rotations. When \(\sigma_1 < \mu < \sigma_2\), the equality holds if and only if \(f\) is \(K_{\Phi,3}\) or one of its rotations. If \(\mu = \sigma_1\), then the equality holds if and only if \(f\) is \(F_\delta\) or one of its rotations. If \(\mu = \sigma_2\), then the equality holds if and only if \(f\) is \(G_\delta\) or one of its rotations.

By making use of Lemma 1.2, we easily obtain the next theorem.

**Theorem 2.2.** Let \(\Phi(z) = 1 + B_1 z + B_2 z^2 + \ldots\) and let \(f(z)\) be in the class \(M_{n,\lambda}^{\alpha,\beta}(\Phi)\). For a complex number \(\mu\) we have:

\[|a_3 - \mu a_2^2| \leq \]
\[ \frac{B_1}{2\lambda(1 + 2\lambda)^n[\alpha + \beta(1 + 2\lambda)]} \max \left\{ 1, \left| \frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1 + \lambda)^{2n}[\alpha + \beta(1 + \lambda)]^2} \right| \right\} . \]

The result is sharp.

References


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