On $X$-Hadamard and $B$-derivations

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Abstract
Let $F$ be an infinite dimensional complex Banach space endowed with a bounded shrinking basis $X$. We seek conditions to relate $X$-Hadamard derivations and $B$-derivations supported on multiplier operators of $F$ relative to $X$. It is seeing that in general the former class is larger than the first and some facts on basis problems are also considered.

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1 Introduction
Throughout this article by $F$ we will denote a complex infinite dimensional Banach space endowed with a bounded shrinking basis $X = \{x_n\}_{n=1}^{\infty}$. Let $F \widehat{\otimes} F^*$ be the tensor product Banach space of $F$ and $F^*$, i.e. the completion of the usual algebraic tensor product with respect to the following cross norm defined for $u \in F \widehat{\otimes} F^*$ as

$$||u||_n = \inf \left\{ \sum_{j=1}^{n} ||x_j|| ||x_j^*|| : u = \sum_{j=1}^{n} x_j \otimes x_j^* \right\}.$$
The space $\mathcal{F} \hat{\otimes} \mathcal{F}^*$ is indeed a Banach algebra under the product so that
\[(x \otimes x^*)(y \otimes y^*) = \langle y, x^* \rangle (x \otimes x^*)\]
if $x, y \in F, x^*, y^* \in F^*$. Then $\mathcal{F} \hat{\otimes} \mathcal{F}^*$ is isometric isomorphic to the Banach algebra $\mathcal{N}_F^*(F)$ of nuclear operators on $F$ (cf. [5], Th. C.1.5, p. 256). This fact allows the transference of the investigation of properties and structure of bounded derivations to a more tractable frame which has essentially the same profile. For previous researches on this matter in a purely algebraic setting, in the frame of Hilbert spaces or on certain Banach algebras of operators the reader can see [1], [2], [3].

The class of bounded derivations on $\mathcal{F} \hat{\otimes} \mathcal{F}^*$ denoted as $\mathcal{D}(\mathcal{F} \hat{\otimes} \mathcal{F}^*)$ becomes a closed subspace of $\mathcal{B}(\mathcal{F} \hat{\otimes} \mathcal{F}^*)$.

Example 1. If $ad_v(u) = u \cdot v - v \cdot u$ for $u, v \in \mathcal{F} \hat{\otimes} \mathcal{F}^*$ then $ad_v \in \mathcal{D}(\mathcal{F} \hat{\otimes} \mathcal{F}^*)$. As usual, $\{ad_v\}_{v \in \mathcal{F} \hat{\otimes} \mathcal{F}^*}$ is the set of inner derivations on $\mathcal{F} \hat{\otimes} \mathcal{F}^*$.

Example 2. Let $\delta_F : \mathcal{B}(F) \to \mathcal{B}(\mathcal{F} \hat{\otimes} \mathcal{F}^*), \delta_F(T) \triangleq \delta_T$ where $\delta_T$ is the unique linear bounded operator on $\mathcal{F} \hat{\otimes} \mathcal{F}^*$ so that $\delta_T(x \otimes x^*) = T(x) \otimes x^* - x \otimes T^*(x^*)$ for all basic tensor $x \otimes x^* \in \mathcal{F} \hat{\otimes} \mathcal{F}^*$. By the universal property on tensor products $\delta_F$ is well defined. Indeed, $\mathcal{R}(\delta_F) \subseteq \mathcal{D}(\mathcal{F} \hat{\otimes} \mathcal{F}^*)$ and $\delta_F \in \mathcal{B}(\mathcal{B}(F), \mathcal{B}(\mathcal{F} \hat{\otimes} \mathcal{F}^*))$.

Example 3. $ad_{x \otimes x^*} = \delta_{x \otimes x^*}$, where as usual $x \otimes x^* \in \mathcal{B}(F)$ denotes the finite rank operator $(x \otimes x^*)(y) = \langle y, x^* \rangle \cdot x$, with $x, y \in F$ and $x^* \in F^*$.

Proposition 1. (cf. [6], [7]) Let $F$ be a Banach space, $\{x_n\}_{n=1}^\infty$ be a shrinking basis of $F$ and let $\{x_n^*\}_{n=1}^\infty$ be its a.s.c.f.. The system of all basic tensor products $x_n \otimes x_m^*$ is basis of $\mathcal{F} \hat{\otimes} \mathcal{F}^*$, arranged into a single sequence as follows: If $m \in \mathbb{N}$ let $n \in \mathbb{N}$ so that $(n - 1)^2 < m \leq n^2$ and then let’s write $x_m = x_{\sigma_1(m)} \otimes x_{\sigma_2(m)}$, with
\[
\sigma(m) = \begin{cases} 
(m - (n - 1)^2, n) & \text{if } (n - 1)^2 + 1 \leq m \leq (n - 1)^2 + n, \\
(n, n^2 - m + 1) & \text{if } (n - 1)^2 + n \leq m \leq n^2.
\end{cases}
\]
Remark 1. In particular, \( \sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) becomes a bijective function. Since \( F^* \hat{\otimes} F \hookrightarrow (F \hat{\otimes} F^*)^* \) we will also write \( z_m^* = x_{\sigma_1(m)}^* \otimes x_{\sigma_2(m)} \), \( m \in \mathbb{N} \). Thus \( \{z_m^*\}_{n=1}^{\infty} \) becomes the a.sc.f. of \( \{z_m\}_{n=1}^{\infty} \).

Theorem 1. (cf. [4]) Let \( F \) be an infinite dimensional Banach space with a shrinking basis \( \{x_n^*\}_{n=1}^{\infty} \). Given \( \delta \in D(F \hat{\otimes} F^*) \) there are unique sequences \( \{h_n\}_{n \in \mathbb{N}} \) and \( \{g_{u,v}^v\}_{u,v \in \mathbb{N}} \) so that if \( u, v \in \mathbb{N} \) then

\[
\delta(z_{\sigma^{-1}(u,v)}) = (h_u - h_v)z_{\sigma^{-1}(u,v)} + \sum_{n=1}^{\infty} (g_u^n \cdot z_{\sigma^{-1}(u,v)} - g_v^n \cdot z_{\sigma^{-1}(u,v)}) .
\]

Indeed, \( h = h[\delta] = (\langle \delta(z_{n^2}), z_{n^2}^* \rangle)_{n \in \mathbb{N}} \) and \( \eta = \eta[\delta] = (\eta_{m,n}^m)_{n,m=1}^{\infty} \), with

\[
\eta_{m}^m = h_{n,1}^{\sigma^{-1}(m,1)} = h_{n}^{m^2} = \begin{cases} 
\langle \delta(z_{n^2}), z_{n^2}^* \rangle & \text{if } n \neq m \\
0 & \text{if } n = m.
\end{cases}
\]

In the sequel we will say that they are the \( h \) and \( g \) sequences of \( \delta \).

Example 4. Let \( \{v_n\}_{n=1}^{\infty} \in \mathbb{C}^\mathbb{N} \) so that \( v = \sum_{n=1}^{\infty} v_n \cdot z_n \) is a well defined element of \( F \hat{\otimes} F^* \). Then \( h[ad_v] = \{v_{n^2-n+1} - v_1\}_{n=1}^{\infty} \), \( \eta_{m,n} [ad_v] = v_{m^2-n+1} \) if \( 1 \leq n < m \) and \( \eta_{m}^m = v_{(n-1)^2+m} \) if \( n > m \).

Definition 1. A derivation \( \delta \in D(F \hat{\otimes} F^*) \) is said to be an \( X \)-Hadamard derivation if its \( g \)-sequence is null.

We will denote the set of all those derivations as \( \mathcal{D}_X(F \hat{\otimes} F^*) \). In [4] it is proved that the former is a complementary Banach subspace of \( D(F \hat{\otimes} F^*) \).

Definition 2. An operator \( \delta \in D(F \hat{\otimes} F^*) \) will be called a \( B \)-derivation if there exists \( T \in \mathcal{B}(F) \) so that \( \delta = \delta_T \) according to the notation of Example 2. We will denote the class of such derivations as \( \mathcal{D}_B(F \hat{\otimes} F^*) \).

Remark 2. Any \( B \)-derivations is infinitely supported because \( \delta_T = \delta_{T+\lambda Id_F} \) if \( T \in \mathcal{F} \) and \( \lambda \in \mathbb{C} \). More precisely, \( \ker(\delta_F = \mathbb{C} \cdot Id_F) \) (see Lemma 1 below).
In Th. 2 will prove that any $X$-Hadamard derivation is a $B$-derivation. In Proposition 2 and Proposition 3 we will analyze necessary and sufficient conditions under which certain natural series of Hadamard derivations are realized as $B$ derivations. It'll then be clear how $h$-sequences determine their structures since the corresponding supports become multiplier operators included by them.

2 X-Hadamard and $B$-derivations

**Lemma 1.** $\ker(\delta_F) = C \cdot Id_F$.

**Proof.** The inclusion $\supseteq$ is evident. Let $T \in B(F)$ so that $\delta_T = 0$ and let $\lambda \in \sigma T$. If $\lambda$ belongs to the compression spectrum of $T$ let $x^* \in F^*-\{0\}$ so that $x^*|_{R(T-\lambda Id_F)} \equiv 0$. For all $x \in F$ we have

$$\langle x, T^*(x^*) \rangle = \langle T(x), x^* \rangle = \langle \lambda x, x^* \rangle = \langle x, \lambda x^* \rangle,$$

i.e. $(t^*-\lambda Id_{F^*})(x^*) = 0$. Moreover, since

$$(T(F) - \lambda x) \otimes x^* = x \otimes (T^*(x^*) - \lambda x^*) = 0,$$

the projective norm is a cross-norm and $x^* \neq 0$ then $T = \lambda Id_F$. If $\lambda \in \sigma_{ap}(T)$ we choose a sequence $\{y_n\}_{n=1}^\infty$ of unit vectors of $F$ so that $T(y_n) - \lambda y_n \rightarrow 0$. If $y^* \in F^*$ then

$$0 = \lim_{n \rightarrow \infty} ||(T(y_n) - \lambda y_n) \otimes y^*||_\pi$$

$$= \lim_{n \rightarrow \infty} ||y_n \otimes (T^*(y^*) - \lambda y^*)||_\pi = ||T^*(y^*) - \lambda y^*||.$$

Reasoning as above we conclude that $T = \lambda Id_F$.

**Lemma 2.** (i) If $r, s \in \mathbb{N}$ then

$$h[\delta_{x_r \odot x_s^*}] = \begin{cases} 
{0, -1, -1, \ldots} & \text{if } r = s = 1, \\
{0, 0, \ldots} & \text{if } r \neq s, \\
e_r & \text{if } r = s > 1.
\end{cases}$$

(ii) If $r \neq s$ then $\eta[\delta_{x_r \odot x_s^*}] = e_s^r$ is the zero matrix elsewhere and has a one in the $(s, r)$ entry. All derivations $\delta_{x_n \odot x_n^*}$ with $n \in \mathbb{N}$ are of Hadamard type.
Proof. (i) If \( r, s, n \in \mathbb{N} \) we get
\[
\delta x_r \otimes x_s^*(x_n \otimes x_m^*) = (x_r \otimes x_s^*) (x_n) \otimes x_m^* - x_n \otimes (x_r \otimes x_s^*)^* (x_m^*) \tag{2}
\]

Letting \( m = 1 \) in (2) then
\[
\delta_{x_r \otimes x_1^*} (x_n \otimes x_1^*) = \sum_{p=1}^{\infty} h_{n,1} \cdot z_p \tag{3}
\]

If \( r = s = 1 \) by (3) is and the first assertion follows. If \( r = s > 1 \) by (3) is \( \delta_{x_r \otimes x_1^*} (x_n \otimes x_1^*) = \delta^n_r \cdot z_{r,2} \) and our third claim follows. Finally, if \( r \neq s = n \) then (3) becomes
\[
\delta_{x_r \otimes x_1^*} (x_n \otimes x_1^*) = z_{r,2} - \delta^n_r \cdot z_{s,2-s+1}
\]

and clearly \( h_{s,}[\delta_{x_r \otimes x_1^*}] = 0 \). If \( s \notin \{r, n\} \) then
\[
\delta_{x_r \otimes x_1^*} (x_n \otimes x_1^*) = -\delta^n_r \cdot z_{s,1}\).

But \( \sigma^{-1}(n, s) = n^2 \) if and only if \( s = 1 \) and as \( r \neq s \) then \( h_{n,}[\delta_{x_r \otimes x_1^*}] = 0 \).

(ii) If \( r, s, n, m \in \mathbb{N} \) and \( n \neq m \) then
\[
\eta_{nm,}[\delta_{x_r \otimes x_1^*}] = \langle \delta X (x_r \otimes x_s^*) (z_{n^2}), z_{m^2} \rangle
\]

\[
= \langle \delta X x_r \otimes x_s^* (x_n \otimes x_1^*), x_m \otimes x_1 \rangle
\]

\[
= \langle \delta^n_r \cdot (x_r \otimes x_1^*), (x_n \otimes (x_r \otimes x_s^*)^* (x_1^*), x_m \otimes x_1 \rangle
\]

\[
= \delta^n_r \cdot \delta^m_r.
\]

The conclusion is now clear.

Remark 3. Given an elementary tensor \( x \otimes x^* \in X \hat{\otimes} X^* \) and \( m \in \mathbb{N} \) we have
\[
\left( \sum_{n=1}^{m} \delta_{x_n \otimes x_n^*} \right) (x \otimes x^*) = \sum_{n=1}^{m} [(x, x_n^*) (x_n \otimes x^*) - (x, x^*) (x_n \otimes x_n^*)]
\]

\[
= \left( \sum_{n=1}^{m} (x, x_n^*) x_n \right) \otimes x^* - x \otimes \left( \sum_{n=1}^{m} (x_n^*, x_n) x_n^* \right)
\]
and so \( \lim_{m \to 0} \left( \sum_{n=1}^{m} \delta_{x_n \otimes x_n^*} \right) (x \otimes x^*) \equiv 0 \). Since the basis \( X \) is assumed to be bounded then \( \rho = \inf_{n,p \in \mathbb{N}} ||x_n||/||x_p^*|| \) is positive (cf. [7], Corollary 3.1, p.20). Consequently, if \( n, m, p \in \mathbb{N} \) and \( n \neq p \) then

\[
\left| \delta_{x_n \otimes x_n^*} \right| \geq \left| \delta_{x_n \otimes x_m^*} \left( \frac{x_m}{||x_m||} \otimes \frac{x_p^*}{||x_p^*||} \right) \right|_\pi = \frac{||x_n||}{||x_m||} \geq \inf_{n \in \mathbb{N}} ||x_n||/ \sup_{m \in \mathbb{N}} ||x_m|| > 0,
\]

i.e. the series \( \sum_{n=1}^{\infty} \delta_{x_n \otimes x_n^*} \) is not convergent.

**Remark 4.** The set \( \{ \delta_{x_n \otimes x_n^*} \}_{n=1}^{\infty} \) is linearly dependent. For, let \( \{c_n\}_{n=1}^{\infty} \) be a sequence of scalars so that

\[
\sum_{n=1}^{\infty} c_n \cdot \delta_{x_n \otimes x_n^*} \equiv 0.
\]

In particular, by (4) is \( \{c_n\}_{n=1}^{\infty} \in c_0 \). If \( r, s \) are two positive integers then

\[
\left[ \sum_{n=1}^{\infty} c_n \cdot \delta_{x_n \otimes x_n^*} \right] (x_r \otimes x_s^*) = (c_r - c_s)(x_r \otimes x_s^*) = 0,
\]

i.e. \( c_r = c_s \). Hence \( \{c_n\}_{n=1}^{\infty} \) becomes the constant zero sequence and the assertion follows.

**Theorem 2.** Every \( X \)-Hadamard derivation is a \( \mathcal{B} \)-derivation.

**Proof.** If \( \delta \in \mathcal{D}(F \otimes F^*) \) and \( x \in F \) the series \( \sum_{n=1}^{\infty} \langle x, x_n^* \rangle \cdot h_n[\delta] \cdot x_n \) converges. For, if \( p, q \in \mathbb{N} \) then

\[
\left\| \sum_{n=p}^{p+q} \langle x, x_n^* \rangle \cdot h_n[\delta] \cdot x_n \right\| \leq \left| \delta \right| \left( \sum_{n=p}^{p+q} \langle x, x_n^* \rangle \cdot x_n \right) \rightarrow 0
\]

i.e. the sequence of corresponding partial sums is a Cauchy sequence. So, it is defined a linear operator \( M_{h[\delta]} : x \rightarrow \sum_{n=1}^{\infty} \langle x, x_n^* \rangle \cdot h_n[\delta] \cdot x_n \) that
is bounded as a consequence of the Banach-Steinhauss theorem. Hence $h[δ] \in M(f^*, X)$, i.e. $h[δ]$ is a multiplier of $F$ relative to the basis $X$. Analogously, if $x^* \in F^*$ the series $\sum_{n=1}^{∞} \langle x_m, x^* \rangle \cdot h[m][δ] \cdot x^*_m$ also converges because if $p, q \in \mathbb{N}$ we get

$$\left\| \sum_{m=p}^{p+q} \langle x_m, x^* \rangle \cdot h[m][δ] \cdot x^*_m \right\| = \left\| \frac{x_1}{\|x_1\|} \otimes \sum_{m=p}^{p+q} \langle x_m, x^* \rangle \cdot h[m][δ] \cdot x^*_m \right\|_{π}$$

$$= \left\| \hat{δ} \left( \frac{x_1}{\|x_1\|} \otimes \sum_{m=p}^{p+q} \langle x_m, x^* \rangle \cdot x^*_m \right) \right\|_{π} = \left\| δ \right\| \left\| \sum_{n=p}^{p+q} \langle x_n, x^* \rangle \cdot x^*_m \right\|.$$ 

It is immediate that $M^*_h[δ](x^*) = \sum_{n=1}^{∞} \langle x_m, x^* \rangle \cdot h[m][δ] \cdot x^*_m$ for all $x^* \in F^*$ and $h[δ]$ is also realizes as a multiplier of $F^*$ relative to the basis $X^*$. Now, if $x \otimes x^*$ is a fixed basic tensor in $F \hat{⊗} F^*$ we can write

$$\delta(x \otimes x^*) = \sum_{n=1}^{∞} \langle x, x_n^* \rangle \sum_{n=1}^{∞} \langle x_m, x^* \rangle (h_n[δ] - h_m[δ]) \cdot (x_n \otimes x_m^*) \cdot x \otimes x^* - x \otimes M^*_h[δ](x^*)$$

and definitely $δ = δ_{M_h[δ]}$.

**Proposition 2.** Let $\{ζ\}_{n=1}^{∞} \in \mathbb{C}^∞$ so that $δ = \sum_{n=1}^{∞} ζ_n \cdot δ_{x_n \otimes x_n^*}$ is a well defined Hadamard derivation.

(i) $h[δ] \in c$, $\{ζ\}_{n=1}^{∞} \in c_0$ and $ζ_m = h[m][δ] - \lim_{n \to ∞} h_n[δ]$ if $m \in \mathbb{N}$.

(ii) $δ = δ_{S}$ where $S \in \mathcal{B}(F)$ is defined for $x \in F$ as

$$S(x) = \sum_{n=1}^{∞} h_n[δ] \cdot \langle x, x_n^* \rangle \cdot x_n - x \cdot \lim_{n \to ∞} h_n[δ].$$

**Proof.**

(i) If $m \in \mathbb{N}$ it is readily seeing that $δ(x_m \otimes x_m^*) = (ζ_m - ζ_1) \cdot zm^2$. Thus $h_1[δ] = 0$ and $h_m[δ] = ζ_m - ζ_1$ if $m > 1$. By (4) we have that $\{ζ\}_{n=1}^{∞} \in c_0$ and we get (ii).
(ii) Let $S_n = \sum_{k=1}^{n} (h_k[h] - \lim_{n \to \infty} x_k \otimes x_k^*), n \in \mathbb{N}$. By the uniform boundedness principle and (ii) the sequence $\{S_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{B}(F)$. Whence, since $S_n(x) \to S(x)$ if $x \in F$ then $S \in \mathcal{B}(F)$. Indeed, if $n, m \in \mathbb{N}$ we get

$$\delta_S(x_n \oplus x_m)^* = ((h_n[h] - \lim_{n \to \infty} h_k[h])x_n) \otimes x_m^* - x_n \otimes ((h_m[h] - \lim_{n \to \infty} h_k[h])x_m^*)$$

$$= (h_n[h] - h_m[h]) \cdot (x_n \otimes x_m^*)$$

$$= \delta(x_n \otimes x_m^*),$$

i.e. $\delta = \delta_S$

**Proposition 3.** Let $\{h_n\}_{n=1}^{\infty} \in M(F, \{x_n\}_{n=1}^{\infty}) \cap M(F^*, \{x_n^*\}_{n=1}^{\infty}) \cap c$ such that $h_1 = 0$. On writing $h_0 \stackrel{\Delta}{=} \lim_{n \to \infty} h_n$ the series $\sum_{n=1}^{\infty} (h_n - h_0) \cdot \delta_{x_n \otimes x_n^*}$ converges to a Hadamard derivation $\delta$ on $\hat{\otimes}F^*$ so that $h[\delta] = \{h_n\}_{n=1}^{\infty}$.

**Proof.** If $S = \sum_{k=1}^{\infty} (h_k - h_0) \cdot x_k \otimes x_k^*$ then $S \in \mathcal{B}(F)$ and

$$\|S\| \leq \|\{h_n\}_{n=1}^{\infty}\|_{M(F, \{x_n\}_{n=1}^{\infty})} + |h_0|.$$

Let $S_n = \sum_{k=1}^{n} (h_k - h_0) \cdot x_k \oplus x_k 6^*, n \in \mathbb{N}$. Given $x \in F$ the sequence $\{S_n(x)\}_{n=1}^{\infty}$ converges because $\{h_n\}_{n=1}^{\infty}$ is a multiplier of $F$ and $\{x_n\}_{n=1}^{\infty}$ is an $F$–complete biorthogonal system. Therefore $\{\|S_n\|\}_{n=1}^{\infty}$ becomes bounded. Now if we fix an elementary tensor $y \otimes y^* \in F \otimes F^*$ and $n \in \mathbb{N}$ then

$$\|(\delta_{S_n} - \delta_S)(y \otimes y^*)\| \leq \left\| \sum_{k>n} (h_k - h_0) \cdot \langle y, x_k^* \rangle \cdot x_k \otimes y^* \right\|$$

$$- y \otimes \sum_{k>n} (h_k - h_0) \cdot \langle x_k, y^* \rangle \cdot x_k^* \right\|_\pi$$

$$\leq \left\| \sum_{k>n} (h_k - h_0) \cdot \langle y, x_k^* \rangle \cdot x_k \right\|_\pi$$

$$+ \|y\| \left\| \sum_{k>n} (h_k - h_0) \cdot \langle x_k, y^* \rangle \cdot x_k^* \right\|_\pi.$$
Since $\{h_n\}_{n=1}^\infty \in M(F, \{x_n\}_{n=1}^\infty) \cap M(F^*, \{x_n^*\}_{n=1}^\infty)$ and $\{x_n\}_{n=1}^\infty$ is a shrinking basis by $(5)$ we see that $\lim_{n \to \infty} (\delta_{S_n} - \delta_S)(y \otimes y^*) = 0$. Indeed, as $F \otimes F^*$ is dense in $F \hat{\otimes} F^*$, $\{S_n\}_{n=1}^\infty$ is bounded and $\|\delta_T\| \leq 2\|T\|$ for all $T \in \mathcal{B}(F)$ then $\delta_T = \sum_{n=1}^\infty (h_n - h_0) \cdot \delta_{x_n \otimes x_n^*}$.

**Problem 1** Giving $T \in \mathcal{B}(F)$ then $\eta[T] = \{(T(x_n), x_m^*)\}_{n,m=1}^\infty$. So it is obvious that $\mathcal{D}_X(F \hat{\otimes} F^*)$. It would be desirable to decide if $\mathcal{D}_B(F \hat{\otimes} F^*)$ is a Banach space.

**Remark 5.** Is $\{\delta_{x_n \otimes x_n^*}\}_{n=1}^\infty$ a basis of $\mathcal{D}_X(F \hat{\otimes} F^*)$? In general this is not the case. For instance, let $F = l^p(N)$ with $1 < p < \infty$ and let $X = \{e_n\}_{n=1}^\infty$, where $e_n = \{\delta_{n,m}\}_{m=1}^\infty$ and $\delta_{n,m}$ the current Kronecker symbol if $n,m \in \mathbb{N}$. Then $X$ is not only a shrinking basis, it is further an unconditional basis of $F$. Consequently, if $T(x) = \sum_{n=1}^\infty \langle x, e_{2n}^* \rangle$ for $x \in F$ then $T \in \mathcal{B}(F)$. It is readily seeing that $\delta_T$ is an $X$-Hadamard derivation. Since $h[\delta_T] = \{0, 1, 0, 1, \ldots\}$ by Prop. 2 $\{\delta_{e_n \otimes e_n^*}\}_{n=1}^\infty$ can not be a basis of $\mathcal{D}_X(F \hat{\otimes} F^*)$.

**Problem 2** Is $\{\delta_{x_n \otimes x_n^*}\}_{n=1}^\infty$ a sequence basis? Can be be constructed a basis of $\mathcal{D}_X((F \hat{\otimes} F^*))$?

**References**


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