Quasiconformality and Compatibility for direct product of bi-Lipschitz homeomorphisms

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Abstract

We continue the research of the consequences of the linear liminf-dilatation used instead of the limsup-dilatation for bi-Lipschitz homeomorphisms. We prove that a direct product \( F = f \times g \) of two homeomorphisms is bi-Lipschitz if and only if \( f \) and \( g \) are bi-Lipschitz. Another result of the paper is that the direct product \( F = f \times g \) is quasiconformal homeomorphism if \( F = f \times g \) is bi-Lipschitz homeomorphism. The converse is true.

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0 Introduction

In this note we shall extend the foundations of the theory of quasiconformal maps on direct products spaces with respect to linear limsup-dilatation, linear liminf-dilatation and bi-Lipschitz homeomorphisms. P. Caraman ([2], pp 127,149,286) have established equivalence the Gehring’s
metric definition and Markushevich-Pesin’s definition in the theory of n-dimensional quasiconformal mappings. We remark that for more details about the evolution of the quasiconformality we shall refer by C. Andreian Cazacu [1].

In [6] and [7] is introduced and developed the quasiconformality by the basis of Markushevich-Pesin’s definition in connection with linear limsup-dilatation.

Let $D$ and $D'$ be a domains in $\mathbb{R}^n$, $F : D \to D'$ is a homeomorphism and

$$d(z, z_0) = |z - z_0| = (|x - x_0|^2 + |y - y_0|^2)^{1/2}$$

the Euclidean distance. For any point $z_0 \in D$ and $t > 0$ such that the ball $\bar{B}(z_0, t) = \{z : d(z, z_0) = |z - z_0| \leq t\}$ be included in $D$, denote

$$L(z_0, F, t) = \max_{d(z, z_0) = t} d(F(z), F(z_0))$$

and

$$l(z_0, F, t) = \min_{d(z, z_0) = t} d(F(z), F(z_0)).$$

J. Väisälä [8] gives that if the linear limsup-dilatation of $F$ at $z_0$

$$H(z_0, F) = \limsup_{t \to 0} \frac{L(z_0, F, t)}{l(z_0, F, t)}$$

is bounded in $D$, i.e. there exists a constant $H < \infty$ such that $H(z_0, F) \leq H$ for every $z_0 \in D$, $F$ is a quasiconformal mapping after the metric definition. In [4], M. Cristea used also in the study of the quasiconformal mappings the linear liminf-dilatation

$$h(z_0, F) = \liminf_{t \to 0} \frac{L(z_0, F, t)}{l(z_0, F, t)}$$

In [5], J. Heinonen and P. Koskela proved that $h(z_0, F) \leq H$ for every $z_0 \in D$ implies that $F$ is quasiconformal and $H(z_0, F) = h(z_0, F)$ a.e., what
increased the importance of $h(z_0, F)$.

Let $U$ and $V$ be domains in $\mathbb{R}^k$ and $\mathbb{R}^m$; $x$ and $y$ arbitrary points in $U$ and $V$, respectively; $f : U \to U' \subset \mathbb{R}^k$, $g : V \to V' \subset \mathbb{R}^m$ be a homeomorphisms; $F = f \times g : U \times V \to U' \times V'$ the direct product of $f$ and $g$; $U \times V$ and $U' \times V'$ being domains in $\mathbb{R}^k \times \mathbb{R}^m$ identified with $\mathbb{R}^n$, $n = k + m$; $z = (x, y)$ is a point in $U \times V$ and $F(z) = (f(x), g(y)) \in U' \times V'$.

Starting from Karmazin’s limsup compatibility condition and Theorem 2\,[6], (for bi-Lipschitz homeomorphisms). We say that $f$ and $g$ are compatible if there is a constant $C$ respectively $c$, such that:

**Condition 1:** $\limsup_{t \to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \leq C$ and $\limsup_{t \to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \leq C$, for any $x_0 \in U$ and $y_0 \in V$, and

**Condition 2:**\,[3] there exists a sequence $t_p \to 0, p \in \mathbb{N}$ such that

$$\frac{L(x_0, f, t_p)}{l(y_0, g, t_p)} \leq c \quad \text{and} \quad \frac{L(y_0, g, t_p)}{l(x_0, f, t_p)} \leq c,$$

for any $x_0 \in U, y_0 \in V$.

**Remark.**\,[3] Condition 2 is fulfilled e.g. if

$$\liminf_{t \to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \leq c \quad \text{and} \quad \limsup_{t \to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \leq C$$

or vice versa, and for case when

$$\liminf_{t \to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \leq c \quad \text{and} \quad \liminf_{t \to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \leq c$$

but there exists a sequence $t_p$ as above.

With these compatibility conditions we succeeded to characterize the quasiconformality and bi-Lipschitz condition of $F$ extending Karmazin’s Theorem 2\,[6]. The main issue is the problem of definition, which follow to be described.
1 Direct products of bi-Lipschitz homeomorphisms

Definition 1.1. A homeomorphism $F : D \rightarrow D'(D, D' \subset \mathbb{R}^n)$ is called \textit{limsup-Lipschitz in domain $D$} if there exists a constant $L, 0 < L < \infty$, such that for almost every points $z_0 \in D$ is satisfied following limsup-Lipschitz condition
\[
\limsup_{t \to 0} \frac{|F(z) - F(z_0)|}{|z - z_0|} \leq L.
\]

The limsup-Lipschitz condition is a classical concept of Lipschitz condition.

Definition 1.2. Let $U$ be an open subset of $\mathbb{R}^n$. A homeomorphism $F : U \rightarrow \mathbb{R}^n$ is said to be \textit{locally limsup-Lipschitz} if for every compact set $A \subset U$ there exists a constant $L_A < \infty$ such that
\[
\limsup_{z \to z_0} \frac{|F(z) - F(z_0)|}{|z - z_0|} \leq L_A
\]
for almost every $z_0 \in A$.

Definition 1.3. A homeomorphism $F : D \rightarrow D'(D, D' \subset \mathbb{R}^n)$ is called \textit{limsup bi-Lipschitz homeomorphism}, if $F$ and $F^{-1}$ are limsup-Lipschitz.

As a variant of Definition 3 with beautiful application is following.

Definition 1.4.

Let $D, D'$ be domains of $\mathbb{R}^n$. A homeomorphism $F : D \rightarrow D'$ is said to be \textit{bi-Lipschitz} if there exists a constant $L, 0 < L < \infty$, such that $F$ satisfies the inequalities
\[
\frac{1}{L} |z - z_0| \leq |F(z) - F(z_0)| \leq L |z - z_0|
\]
for every $z_0 \in D$ and for $|z - z_0|$ sufficiently small.

The result was first proved by Karmazin, here we shell it by a different method.

Theorem 1.5. ([6]) A homeomorphism $F = f \times g$ is limsup bi-Lipschitz in domain $U \times V$ if and only if its homeomorphisms $f$ and $g$ are also limsup bi-Lipschitz in domains $U$ and $V$, respectively.
Proof. Necessity. Let $F : U \times V \to U' \times V'$ be a limsup bi-Lipschitz homeomorphism at the point $z_0 \in U \times V$ if satisfies the condition

$$\frac{|z - z_0|}{L} \leq |F(z) - F(z_0)| \leq L \cdot |z - z_0|,$$

where $L < \infty$.

Consider the point $x_0 \in U$ such that for some $y_0 \in V$, $F(z)$ satisfies limsup-Lipschitz condition in point $z_0 = (x_0, y_0)$ with a constant $L < \infty$.

Let $U_0$ be the set of all $x_0$, then $mes U_0 = mes U$.

Indeed,

$U \setminus U_0 = \{x_0 \in U :$ for which does not exists $y_0 \in V$ such that $z_0 = (x_0, y_0)$ satisfies Definition 1.1$\}$.

Then

$(U \setminus U_0) \times V = \{(x_0, y_0) : x_0 \in U \setminus U_0, y_0 \in V$ such that at $z_0 = (x_0, y_0)$ does not satisfy Definition 1.1 $\} \subset \{(x, y) \in U \times V : (x, y)$

does not satisfy Definition 1.1$\}$,

has measure zero. By inclusion $n$-mes $((U \setminus U_0) \times V) = 0$.

By Theorem I ([89], p.153), $((U \setminus U_0) \times V)$ is $n$-measurable.

Thus

$$n\text{-mes}((U \setminus U_0) \times V) = (k\text{-mes} (U \setminus U_0)) \cdot (m\text{-mes} V) = 0.$$

Hence

$$k\text{-mes} (U \setminus U_0) = 0 \Rightarrow k\text{-mes} U_0 = k\text{-mes} U.$$

Let $x \in U_0$ and $\varepsilon > 0$, arbitrary. Then, for each point $z = (x, y_0)$ we have $|z - z_0| = |x - x_0|$.

$$|f(x) - f(x_0)| \leq |F(z) - F(z_0)| \leq (L + \varepsilon)|z - z_0| = (L + \varepsilon)|x - x_0|,$$

$$|f(x) - f(x_0)| \leq (L + \varepsilon)|x - x_0|.$$
Then clearly the homeomorphism \( f(x) \) satisfies the limsup-Lipschitz condition at point \( x_0 \). Similarly, we can prove for the homeomorphism \( g(y) \).

Let \( y \in V \) and \( \epsilon > 0 \), arbitrary.

Then, for each point \( z = (x_0, y) \) we have \(|z - z_0| = |y - y_0|\), therefore \(|g(y) - g(y_0)| \leq (L + \epsilon)|y - y_0|\).

Hence, \( g(y) \) is limsup-Lipschitz homeomorphism. We also, \( f^{-1} \) and \( g^{-1} \) satisfies the limsup-Lipschitz condition. Consequently, the homeomorphisms \( f \) and \( g \) are the limsup bi-Lipschitz.

**Sufficiency:** Let \( f \) and \( g \) be a limsup bi-Lipschitz homeomorphisms, then we show that \( F \) is limsup bi-Lipschitz homeomorphism. With this aim we first suppose that \( f(x) \) at the point \( x_0 \in U \) satisfies the limsup-Lipschitz condition with a constant \( L \) and \( g(y) \) at the point \( y_0 \in V \) satisfies the limsup-Lipschitz condition with constant \( L \).

If \( \epsilon > 0 \) is a positive number arbitrary and for \( |x - x_0|, |y - y_0| \) sufficiently small, we have

\[ |f(x) - f(x_0)| \leq (L + \epsilon)|x - x_0|, \quad |g(y) - g(y_0)| \leq (L + \epsilon)|y - y_0|. \]

Hence

\[ |F(z) - F(z_0)| = |(f(x), g(y)) - (f(x_0), g(y_0))| \leq |f(x) - f(x_0)| + |g(y) - g(y_0)| \leq (L + \epsilon)|x - x_0| + (L + \epsilon)|y - y_0| \leq 2(L + \epsilon)|z - z_0|. \]

These proves that \( F(z) \) satisfies the limsup-Lipschitz condition with constant \( 2L \). Obviously, \( F^{-1}(z) = f^{-1}(x) \times g^{-1}(y) \) satisfies the limsup-Lipschitz condition. Hence, \( F(z) \) is the bi-Lipschitz homeomorphism.

**Theorem 1.6** ([6], p.31) Let \( U, U' \subset \mathbb{R}^k \) and \( V, V' \subset \mathbb{R}^m \) be a domains. If \( f : U \to U' \) and \( g : V \to V' \) be a limsup-compatible homeomorphisms in domain \( D = U \times V \), then \( f \) and \( g \) are limsup bi-Lipschitz homeomorphisms in domains \( U \) and \( V \), respectively.
Proof. Suppose that $f$ and $g$ be a limsup-compatible homeomorphisms in the domain $D$. Then, there exist a constant $C < \infty$, such that for almost every points $z_0 \in U \times V$ we have

$$\limsup_{t \to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \leq C \quad \text{and} \quad \limsup_{t \to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \leq C,$$

where $L(x_0, f, t) = \max_{x \in \partial B^k(x_0, t)} |f(x) - f(x_0)|$, $l(x_0, f, t) = \min_{x \in \partial B^k(x_0, t)} |f(x) - f(x_0)|$, then for $t > 0$ sufficient to small $B^k(x_0, t) \subset U$; $L(y_0, g, t) = \max_{y \in \partial B^m(y_0, t)} |g(y) - g(y_0)|$, $l(y_0, g, t) = \min_{y \in \partial B^m(y_0, t)} |g(y) - g(y_0)|$, then for $t > 0$ sufficient small $B^m(y_0, t) \subset V$.

The role of neighbourhoods $U$ and $V$ is taken the balls $B^k(x_0, t)$ and $B^m(y_0, t)$, respectively.

We shown that there exist a constant $L$ such that for almost every point $x_0$ in $U$ and respectively $y_0$ in $V$ we have

$$\limsup_{t \to 0} \frac{L(x_0, f, t)}{t} \leq L \quad \text{and} \quad \limsup_{t \to 0} \frac{L(y_0, g, t)}{t} \leq L,$$

is it implies that $f$ and $g$ are limsup bi-Lipschitz.

By Corollary 13[3] the $f$ and $g$ be quasiconformal homeomorphisms. Using Theorem 1.5[3] are differentiable almost every in domains $U$ and $V$, respectively

Then, we to choose $i \in \mathbb{N}$, such that for

$$A_i = \left\{ x_0 \in U : \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq i \right\},$$

mes $A_i > 0$.

Let us show that there exist the $i < \infty$.

Indeed, if there exist a $x_0$ be a point of differentiable and is satisfying inequality:

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq i(x_0).$$
For each $i \in \mathbb{N}$ to find $A_i \subset A_{i+1}$, therefore $\bigcup_{i \in \mathbb{N}} A_i \subset U$

and $\bigcup_{i \in \mathbb{N}} A_i$ included the set of points in $U$ for where $f$ is differentiable, we have measure equal with measure of $U$. Hence $\text{mes} \left( \bigcup_{i \in \mathbb{N}} A_i \right) = \text{mes} U$, which implies that there exist $i \in \mathbb{N}$, with $\text{mes} A_i > 0$.

Similarly, we choose $j < \infty$, such that for $B_j = \{ y_0 \in V : \limsup_{y \to y_0} \frac{|g(y) - g(y_0)|}{|y - y_0|} \leq j \}$, $\text{mes} B_j > 0$.

Because $y_0$ is a point of differentiability, this establishes that

$$\limsup_{y \to y_0} \frac{|g(y) - g(y_0)|}{|y - y_0|} \leq j(y_0).$$

For all $j \in \mathbb{N}$ we find $B_j \subset B_{j+1}$, therefore $\bigcup_{j \in \mathbb{N}} B_j \subset V$ and $\bigcup_{j \in \mathbb{N}} B_j$ included all the points by $V$ which $g$ is differentiable, we have measure equal with measure of $V$. Hence $\text{mes} \bigcup_{j \in \mathbb{N}} B_j = \text{mes} V$, this implies that there exist $j \in \mathbb{N}$ with $\text{mes} V_j > 0$.

This implies that exist the point $y_0 \in V$ such that for almost every points $x \in U$ satisfies limsup-compatibility condition in $(x, y_0)$.

We show that exist a point $y_0$. For every point $y \in V$, we consider

$$U_y = \{ x \in U : \text{Condition 1 is satisfies in } (x, y) \}$$

și

$$D_0 = \{ z \in D : \text{Condition 1 is satisfies in } (x, y) \}.$$

Evidently, $D_0 = \bigcup_{y \in V} \{ U_y \times \{ y \} \} \subset D$. By hypotheses, $\text{mes} D_0 = \text{mes} D$ and from Fubini’s Theorem ([89], p.156-159) we have

$$\text{mes} D_0 = \int_V \text{mes} U_y dy \leq \text{mes} U \cdot \text{mes} V = \text{mes} D,$$
where \[ \int_V \mes U_y dy = \mes U \cdot \mes V. \]

We denote by \( A = \{ y \in V : \mes U_y < \mes U \} \) and \( \mes A = \{ y \in V : \mes U_y < \mes U \} = 0. \) Hence \( \mes A = 0 \) the implies for almost every points \( y \in V, \mes U_y = \mes U, \) therefore

\[ \mes V = \mes V \setminus A \] where \( V \setminus A = \{ y \in V : \mes U_y = \mes U \}. \)

There exist a point \( y_0 \in V \setminus A \) with \( \mes U_{y_0} = \mes U, \) hence we have that for almost every points \( x \in U, \) Condition 1 is satisfies in point \( (x, y_0). \)

We show that, if \( x_0 \in A \cap U_{y_0} \) and for any \( x \in U \) we have

\[ \limsup_{t \to 0} \frac{L(x, f, t)}{L(x_0, f, t)} \leq C^2 \]

\[ \frac{L(x, f, t)}{L(x_0, f, t)} \leq \frac{L(x, f, t)}{l(y_0, g, t)} \cdot \frac{l(y_0, g, t)}{L(x_0, f, t)} \leq \frac{L(x, f, t)}{L(x_0, f, t)} \leq C^2. \]

By \( \frac{L(x, f, t)}{L(x_0, f, t)} \leq C^2 \) and \( \limsup_{t \to 0} \frac{L(x_0, f, t)}{t} \leq i, \) we obtain

\[ L(x, f, t) \leq C^2 L(x_0, f, t) \]

\[ \frac{L(x, f, t)}{t} \leq C^2 \frac{L(x_0, f, t)}{t}, \]

\[ \limsup_{t \to 0} \frac{L(x, f, t)}{t} \leq C^2 \limsup_{t \to 0} \frac{L(x_0, f, t)}{t} \leq C^2 i, \]

\[ \limsup_{t \to 0} \frac{L(x, f, t)}{t} \leq C^2 i. \]

Therefore, \( f(x) \) for almost every points in \( U \) satisfies limsup-Lipschitz condition with a constant \( L = C^2 i. \)

Similarly, we show for almost every point \( y \) in \( V, \) such that the homeomorphism \( g(y) \) satisfies limsup-Lipschitz condition with constant \( L = C^2 j. \)

Mappings \( f^{-1} \) and \( g^{-1} \) are limsup-compatibly in domain \( U' \times V' \) and with
above reasoning, also limsup-Lipschitz condition is satisfies. Therefore, 
\( f(x) \) and \( g(y) \) are bi-Lipschitz in domains \( U, V \) respectively.

**Proposition 1.7** Let \( f : U \to \mathbb{R}^k(U \subset \mathbb{R}^k) \) be a homeomorphism. If \( f \) is 
\( L\)-bi-Lipschitz homeomorphism, then \( f \) is \( L^2\)-quasiconformal.

**Proof.** By \( f : U \to U' \) bi-Lipschitz for almost all point \( x_0 \) from \( U \), we have

\[
\frac{1}{L} \leq \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq L.
\]

Denoted \( |z - z_0| = t \), we can may write

\[
\frac{t}{L} \leq l(x_0, f, t) \leq L(x_0, f, t) \leq Lt,
\]

for almost every in \( U \), which implies

\[
\frac{L(x_0, f, t)}{l(x_0, f, t)} \leq \frac{Lt}{t} = L^2.
\]

By limsup-quasiconformal Condition, we have \( f \) is \( L^2\)-quasiconformal homeomorphism. In particular, this result is applied and for \( F = f \times g \) with the above condition.

**Theorem 1.8** If \( f \) and \( g \) are bi-Lipschitz homeomorphisms, then \( f \) and \( g \) are compatible homeomorphisms

**Proof.** Suppose that \( f \) and \( g \) are bi-Lipschitz, by Theorem 4.1.5 we have that \( F \) is bi-Lipschitz, from Propozition 4.1.7 we have that \( F \) is quasiconformal, using Theorem 3.3.5 \( f \) and \( g \) are compatibles.

**Corollary 1.9** In the above conditions for homeomorphism \( F = f \times g \), we have

\[
F = f \times g \quad \iff \quad f \text{ and } g \quad \iff \quad f \text{ and } g \quad \iff \quad F = f \times g
\]

homeomorphism limsup— bi — Lipschitz bi — Lipschitz

limsup— compatible

quasiconformal
2 The relation between liminf-compatible condition and bi-Lipschitz condition of direct products of homeomorphisms

Theorem 2.1 Let $f : U \to \mathbb{R}^k (U \subset \mathbb{R}^k)$ and $g : V \to \mathbb{R}^m (V \subset \mathbb{R}^m)$ be a liminf-compatible homeomorphisms in domain $D = U \times V \subset \mathbb{R}^k \times \mathbb{R}^m$. Suppose that there is $x_1, x_2 \in U$ and $y_1, y_2 \in V$ such that

\[ \liminf_{t \to 0} \frac{L(y_1, g, t)}{t} > 0 \quad \text{and} \quad \limsup_{t \to 0} \frac{l(y_2, g, t)}{t} < \infty, \]

\[ \liminf_{t \to 0} \frac{L(x_1, f, t)}{t} > 0 \quad \text{and} \quad \limsup_{t \to 0} \frac{l(x_2, f, t)}{t} < \infty. \]

Then $f$ and $g$ are bi-Lipschitz homeomorphisms in domains $U$ and $V$, respectively.

Proof. Let $f$ and $g$ be a liminf-compatible at the point $z = (x, y) \in D$. By compatibility, Condition 2, there is a constant $c < \infty$ and a sequence common $t_p \to 0, p \in \mathbb{N}$, such that conditions

\[ \frac{L(x, f, t_p)}{l(y, g, t_p)} \leq c \quad \text{and} \quad \frac{L(y, g, t_p)}{l(x, f, t_p)} \leq c \]

are fulfilled.

Step 1.

a) Suppose that there is a constant $l_2 > 0$ such that

\[ \liminf_{t \to 0} \frac{L(y_1, g, t)}{t} > l_2 > 0; \]

then for $t$ sufficient to small, we have $L(y, g, t) > l_2 t$.

b) Suppose that there is a constant $l_1 > 0$ such that

\[ \liminf_{t \to 0} \frac{l(x, f, t)}{t} \geq l_1 > 0 \]

for all $x \in U$. 

Indeed, if assertion is false, then there is \( x_k \in U \), for every \( k \in \mathbb{N} \) with
\[
\liminf_{t \to 0} \frac{l(x_k, f, t)}{t} < \frac{1}{k}.
\]
We can take a sequence \( \tilde{t}_p(x_k) \to 0, p \in \mathbb{N} \), such that
\[
l(x_k, f, \tilde{t}_p(x_k)) < \frac{\tilde{t}_p(x_k)}{k}
\]
and
\[
L(y_1, g, \tilde{t}_p(x_k)) > l_2 \tilde{t}_p(x_k).
\]
By compatibility, Condition 2, we obtain
\[
kl_2 = \frac{l_2 \tilde{t}_p(x_k)}{\tilde{t}_p(x_k)} < \frac{L(y_1, g, \tilde{t}_p(x_k))}{l(x_k, f, \tilde{t}_p(x_k))} \leq c.
\]
Because \( k \to \infty \) and \( 0 < c < \infty \), we obtain a contradiction. Let now \( x \in U \).

There exists a constant \( l_1 > 0 \), such that inequality is true
\[
\liminf_{t \to 0} \frac{l(x, f, t)}{t} \geq l_1 > 0
\]
for every \( x \in U \).

Suppose first that \( x_0 \in U \) for which we have
\[
\liminf_{t \to 0} \frac{l(x_0, f, t)}{t} \geq l_1 \text{ implies that}
\]
\[
l(x_0, f, t) \geq l_1 t, \text{ for } 0 < t < t'_{x_0}.
\]
Let \( x \in B(x_0, t'_{x_0}), \ | x - x_0 | = t \leq t'_{x_0} \) and \( l(x_0, f, t) \geq l_1 t = l_1 | x - x_0 | \).

Because
\[
l(x_0, f, t) \leq \min_{|x - x_0| = t} | f(x) - f(x_0) | \leq | f(x) - f(x_0) |,
\]
to obtain
\[
l_1 | x - x_0 | \leq | f(x) - f(x_0) |, \text{ for all } x \in B(x_0, t'_{x_0}).
\]
We proved (1) for every $x_0 \in U$.

**Step 2**.

a) Suppose that is satisfied a condition

$$\limsup_{t \to 0} \frac{l(y_2, g, t)}{t} < \infty,$$

for $t$ sufficiently small and a constant $0 < l_2 < \infty$, such that we have

$$l(y_2, g, t) \leq tl_2.$$

b) We assume that exists a constant $L_1 > 0$, such that

$$\limsup_{t \to 0} \frac{L(x, f, t)}{t} < L_1,$$

for all $x \in U$.

Indeed, if assertion is false therefore does not exist a constant $L_1 > 0$, then for every $k \in \mathbb{N}$, exists $x_k^* \in U$, such that

$$\limsup_{t \to 0} \frac{L(x_k^*, f, t)}{t} \geq k.$$

For every $k \in \mathbb{N}$ we can find a sequence $t_p(x_k^*)$ with

$$L(x_k^*, f, t_p(x_k^*)) \geq kt_p(x_k^*)$$

and

$$l(y_2, g, t_p(x_k^*)) \leq l_2t_p(x_k^*).$$

By compatibility, Condition 2, we have

$$\frac{k}{l_2} \leq \frac{L(x_k^*, f, t_p(x_k^*))}{l(y_2, g, t_p(x_k^*))} < c,$$

because $k \to \infty$, $\frac{k}{l_2} \to \infty$, we reach a contradiction.

Hence, there is a constant $L_1 > 0$ such that $L(x, f, t) < L_1 t$, for all $x \in U$.

Let now $x_0 \in U$, for which we have

$$L(x_0, f, t) \leq L_1 t, \text{ for } 0 < t < t_{x_0}''.$$
Let $x \in B(x_0, t''_{x_0})$, $|x - x_0| = t \leq t''_{x_0}$, then

$$L(x_0, f, t) = \max_{|x-x_0|=t} |f(x) - f(x_0)| \leq L_1 |x - x_0|,$$

therefore

$$|f(x) - f(x_0)| \leq L_1 |x - x_0|.$$

Suppose that $t_{x_0} = \min\{t'_{x_0}, t''_{x_0}\}$. By (1) and (2) we obtain

$$l_1 |x - x_0| \leq |f(x) - f(x_0)| \leq L_1 |x - x_0|$$

for all $x \in B(x_0, t_{x_0})$ and for all $x_0 \in U$.

By the condition $i)$ and compatibility Condition 2 we proved that $f$ is bi-Lipschitz homeomorphism. Similarly, by $ii)$ and compatibility condition we showed that $g$ is a bi-Lipschitz. The theorem is proved.

**Remark.** This theorem is valid and in case limsup-compatibility Condition 1, without not even a supplementary hypothesis.

**References**


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