Steepest descent approximations in Banach space\textsuperscript{1}

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Abstract

Let $E$ be a real Banach space and let $A : E \to E$ be a Lipschitzian generalized strongly accretive operator. Let $z \in E$ and $x_0$ be an arbitrary initial value in $E$ for which the steepest descent approximation scheme is defined by

\begin{align*}
x_{n+1} &= x_n - \alpha_n (Ay_n - z), \\
y_n &= x_n - \beta_n (Ax_n - z), \quad n = 0, 1, 2 \ldots,
\end{align*}

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

\begin{itemize}
  \item[(i)] $0 \leq \alpha_n, \beta_n \leq 1,$
  \item[(ii)] $\sum_{n=0}^{\infty} \alpha_n = +\infty,$
  \item[(iii)] $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n,$
\end{itemize}

converges strongly to the unique solution of the equation $Ax = z$.

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1 Introduction

Let $E$ be a real Banach space and let $E^*$ be its dual space. The normalized
duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{ u \in E^* : \langle x, u \rangle = \| x \| \| u \| , \| u \| = \| x \| \},$$

where $\langle \ldots \rangle$ denotes the generalized duality pairing.

A mapping $A$ with domain $D(A)$ and range $R(A)$ in $E$ is said to be
strongly accretive if there exist a constant $k \in (0, 1)$ such that for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k \| x - y \|^2,$$

and is called $\phi$-strongly accretive if there is a strictly increasing function
$\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for any $x, y \in D(A)$ there exist $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \phi(\| x - y \|) \| x - y \|.$$

The mapping $A$ is called generalized $\Phi$-accretive if there exist a strictly
increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for all $x, y \in D(A)$ there exist $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \Phi(\| x - y \|).$$

It is well known that the class of generalized $\Phi$-accretive mappings in-
cludes the class of $\phi$-strongly accretive operators as a special case (one set
$\Phi(s) = s \phi(s)$ for all $s \in [0, \infty)$).

Let $N(A) := \{ x \in D(A) : Ax = 0 \} \neq \emptyset$.

The mapping $A$ is called strongly quasi-accretive if there exist $k \in (0, 1)$
such that for all $x \in D(A), p \in N(A)$ there exist $j(x - p) \in J(x - p)$ such that

$$\langle Ax - Ap, j(x - p) \rangle \geq k \| x - p \|^2.$$
A is called φ-strongly quasi-accretive if there exist a strictly increasing function \( \phi : [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \) such that for all \( x \in D(A) \), \( p \in N(A) \) there exist \( j(x-p) \in J(x-p) \) such that
\[
\langle Ax - Ap, j(x-p) \rangle \geq \phi(\|x-p\|) \|x-p\|.
\]

Finally, \( A \) is called generalized Φ-quasi-accretive if there exist a strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \) such that for all \( x \in D(A) \), \( p \in N(A) \) there exist \( j(x-p) \in J(x-p) \) such that
\[
\langle Ax - Ap, j(x-p) \rangle \geq \Phi(\|x-p\|) \|x-p\|.
\]

A mapping \( G : E \to E \) is called Lipschitz if there exists a constants \( L > 0 \) such that \( \|Gx - Gy\| \leq L\|x-y\| \) for all \( x, y \in D(G) \). Closely related to the class of accretive-type mappings are those of pseudo-contractive types.

A mapping \( T : E \to E \) is called strongly pseudo-contractive if and only if \( I-T \) is strongly accretive, and is called strongly \( \phi \)-pseudo-contractive if and only if \( (I - T) \) is \( \phi \)-strongly accretive. The mapping \( T \) is called generalized \( \Phi \)-pseudo-contractive if and only if \( (I - T) \) is generalized \( \Phi \)-accretive.

In [5, page 9], Ciric et al. showed by taking an example that a generalized \( \Phi \)-strongly quasi-accretive operator is not necessarily a \( \phi \)-strongly quasi-accretive operator.

If \( F(T) := \{ x \in E : T x = x \} \neq \emptyset \), the mapping \( T \) is called strongly hemi-contractive if and only if \( (I - T) \) is strongly quasi-accretive; it is called \( \phi \)-hemi-contractive if and only if \( (I - T) \) is \( \phi \)-strongly quasi-accretive; and \( T \) is called generalized \( \Phi \)-hemi-contractive if and only if \( (I - T) \) is generalized \( \Phi \)-quasi-accretive.

The class of generalized \( \Phi \)-hemi-contractive mappings is the most general (among those defined above) for which \( T \) has a unique fixed point. The relation between the zeros of accretive-type operators and the fixed points of pseudo-contractive-type mappings is well known [1,8,11].
The steepest descent approximation process for monotone operators was introduced independently by Vainberg [13] and Zarantonello [15]. Mann [9] introduced an iteration process which, under suitable conditions, converges to a zero in Hilbert space. The Mann iteration scheme was further developed by Ishikawa [6]. Recently, Ciric et al. [5], Zhou and Guo [16], Morales and Chidume [12], Chidume [3], Xu and Roach [14] and many others have studied the characteristic conditions for the convergence of the steepest descent approximations.

Morales and Chidume proved the following theorem:

**Theorem 1.** Let $X$ be a uniformly smooth Banach space and let $T : X \to X$ be a $\phi$-strongly accretive operator, which is bounded and demicontinuous. Let $z \in X$ and let $x_0$ be an arbitrary initial value in $X$ for which $\lim_{t \to \infty} \inf \phi(t) > \|Tx_0\|$. Then the steepest descent approximation scheme

$$x_{n+1} = x_n - (Tx_n - z), \ n = 0, 1, 2 \ldots,$$

converges strongly to the unique solution of the equation $Tx = z$ provided that the sequence $\{\alpha_n\}$ of positive real numbers satisfies the following:

(i) $\{\alpha_n\}$ is bounded above by some fixed constant,

(ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$,

(iii) $\sum_{n=0}^{\infty} \alpha_n b(\alpha_n) < +\infty$,

where $b : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function.

In [5], Ciric et al. proved the following theorem:

**Theorem 2.** Let $X$ be a uniformly smooth Banach space and let $T : X \to X$ be a bounded and demicontinuous generalized strongly accretive operator. Let $z \in X$ and let $x_0$ be an arbitrary initial value in $X$ for which $\|Tx_0\| < $
Then a steepest descent approximation scheme defined by
\[ x_{n+1} = x_n - \alpha_n(Ty_n - z), \ n = 0, 1, 2, \ldots, \]
\[ y_n = x_n - \beta_n(Tx_n - z), \ n = 0, 1, 2, \ldots, \]
where the sequence \( \{\alpha_n\} \) of positive real numbers satisfies the following conditions:

(i) \( \alpha_n \leq \lambda \), where \( \lambda \) is some fixed constant,

(ii) \( \sum_{n=0}^{\infty} \alpha_n = +\infty \),

(iii) \( \alpha_n \to 0 \) as \( n \to \infty \), converges strongly to the unique solution of the equation \( Tx = z \).

The purpose of this paper is to continue a study of sufficient conditions for the convergence of the steepest descent approximation process to the zero of a generalized strongly accretive operator. We also extend and improve the results which include the steepest descend method considered by Ciric et al. [5], Morales and Chidume [12], Chidume [3] and Xu and Roach [14] for a bounded \( \phi \)-strongly quasi-accretive operator and also the generalized steepest descend method considered by Zhou and Guo [16] for a bounded \( \phi \)-strongly quasi-accretive operator.

2 Main results

The following lemmas are now well known.

**Lemma 1.** [2] Let \( J : E \to 2^E \) be the normalized duality mapping. Then for any \( x, y \in E \), we have
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \text{ for all } j(x + y) \in J(x + y). \]

Suppose there exist a strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \).
Lemma 2. [10] Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\Phi(0) = 0$ and $\{a_n\}, \{b_n\}, \{c_n\}$ be nonnegative real sequences such that
\[
\lim_{n\to\infty} b_n = 0, \quad c_n = o(b_n), \quad \sum_{n=0}^{\infty} b_n = \infty.
\]
Suppose that for all $n \geq 0$,
\[
a_{n+1}^2 \leq a_n^2 - \Phi(a_{n+1}) b_n + c_n,
\]
then $\lim_{n\to\infty} a_n = 0$.

Theorem 3.. Let $E$ be a real Banach space and let $A : E \to E$ be a Lipschitzian generalized strongly accretive operator. Let $z \in E$ and $x_0$ be an arbitrary initial value in $E$ for which the steepest descent approximation scheme is defined by
\[
x_{n+1} = x_n - \alpha_n (Ay_n - z),
\]
\[
y_n = x_n - \beta_n (Ax_n - z), \quad n = 0, 1, 2, \ldots,
\]
where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:
1. $0 \leq \alpha_n, \beta_n \leq 1$,
2. $\sum_{n=0}^{\infty} \alpha_n = +\infty$,
3. $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$,

converges strongly to the unique solution of the equation $Ax = z$.

Proof. Following the technique of Chidume and Chidume [4], without loss of generality we may assume that $z = 0$. Define by $p$ the unique zero of $A$.

By $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$, imply there exist $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\alpha_n \leq \delta$ and $\beta_n \leq \delta'$;
\[
0 < \delta = \min \left\{ \frac{1}{3L}, \frac{\Phi(2\Phi^{-1}(a_0))}{36L^2 [\Phi^{-1}(a_0)]^2} \right\},
\]
\[
0 < \delta' = \min \left\{ \frac{1}{2L}, \frac{\Phi(2\Phi^{-1}(a_0))}{24L^2 [\Phi^{-1}(a_0)]^2} \right\}.
\]
Define \( a_0 := \|Ax_{n_0}\| \|x_{n_0} - p\| \). Then from (1), we obtain that \( \|x_{n_0} - p\| \leq \Phi^{-1}(a_0) \).

By induction, we shall prove that \( \|x_n - p\| \leq 2\Phi^{-1}(a_0) \) for all \( n \geq n_0 \). Clearly, the inequality holds for \( n = n_0 \). Suppose it holds for some \( n \geq n_0 \), i.e., \( \|x_n - p\| \leq 2\Phi^{-1}(a_0) \). We prove that \( \|x_{n+1} - p\| \leq 2\Phi^{-1}(a_0) \). Suppose that this is not true. Then \( \|x_{n+1} - p\| > 2\Phi^{-1}(a_0) \), so that \( \Phi(\|x_{n+1} - p\|) > \Phi(2\Phi^{-1}(a_0)) \). Using the recursion formula (2), we have the following estimates

\[
\|Ax_n\| = \|Ax_n - Ap\| \leq L \|x_n - p\| \leq 2L\Phi^{-1}(a_0),
\]

\[
\|y_n - p\| = \|x_n - p - \beta_n Ax_n\| \leq \|x_n - p\| + \beta_n \|Ax_n\|
\leq 2\Phi^{-1}(a_0) + 2L\Phi^{-1}(a_0)\beta_n \leq 3\Phi^{-1}(a_0),
\]

\[
\|x_{n+1} - p\| = \|x_n - p - \alpha_n Ay_n\| \leq \|x_n - p\| + \alpha_n \|Ay_n\|
\leq \|x_n - p\| + L\alpha_n \|y_n - p\|
\leq 2\Phi^{-1}(a_0) + 3L\Phi^{-1}(a_0)\alpha_n \leq 3\Phi^{-1}(a_0).
\]

With these estimates and again using the recursion formula (2), we obtain by Lemma 1 that

\[(3) \quad \|x_{n+1} - p\|^2 = \|x_n - p - \alpha_n Ay_n\|^2
\leq \|x_n - p\|^2 - 2\alpha_n \langle Ay_n, j(x_{n+1} - p) \rangle
= \|x_n - p\|^2 - 2\alpha_n \langle Ax_{n+1}, j(x_{n+1} - p) \rangle
+ 2\alpha_n \langle Ax_{n+1} - Ay_n, j(x_{n+1} - p) \rangle
\leq \|x_n - p\|^2 - 2\alpha_n \Phi(\|x_{n+1} - p\|)
+ 2\alpha_n \|Ax_{n+1} - Ay_n\| \|x_{n+1} - p\|
\leq \|x_n - p\|^2 - 2\alpha_n \Phi(\|x_{n+1} - p\|)
+ 2\alpha_n L \|x_{n+1} - y_n\| \|x_{n+1} - p\| ,
\]

where

\[
\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - y_n\| = \alpha_n \|Ay_n\| + \beta_n \|Ax_n\|
\leq L\alpha_n \|y_n - p\| + L\beta_n \|x_n - p\| \leq L\Phi^{-1}(a_0)(3\alpha_n + 2\beta_n),
\]
and consequently from (3), we get

\[
(4) \quad \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - 2\alpha_n \Phi(\|x_{n+1} - p\|) + 2L^2 [\Phi^{-1}(a_0)]^2 (3\alpha_n^2 + 2\alpha_n\beta_n) \|x_{n+1} - p\| \leq \|x_n - p\|^2 - 2\alpha_n \Phi(2\Phi^{-1}(a_0)) + 6L^2 \left[\Phi^{-1}(a_0)\right]^2 (3\alpha_n^2 + 2\alpha_n\beta_n) \\
\]

Thus

\[
\alpha_n \Phi(2\Phi^{-1}(a_0)) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2, 
\]

implies

\[
\Phi(2\Phi^{-1}(a_0)) \sum_{n=n_0}^{j} \alpha_n \leq \sum_{n=n_0}^{j} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) = \|x_{n_0} - p\|^2, 
\]

so that as \(j \to \infty\) we have

\[
\Phi(2\Phi^{-1}(a_0)) \sum_{n=n_0}^{\infty} \alpha_n \leq \|x_{n_0} - p\|^2 < \infty, 
\]

which implies that \(\sum_{n=0}^{\infty} \alpha_n < \infty\), a contradiction. Hence, \(\|x_{n+1} - p\| \leq 2\Phi^{-1}(a_0)\); thus \(\{x_n\}\) is bounded. Consider

\[
\|y_n - x_n\| = \|x_n - \beta_n Ax_n - x_n\| = \beta_n \|Ax_n\| \leq L\beta_n \|x_n - p\| \leq 2L\Phi^{-1}(a_0)\beta_n \to 0 \text{ as } n \to \infty, 
\]

implies the sequence \(\{y_n - x_n\}\) is bounded. Since \(\|y_n - p\| \leq \|y_n - x_n\| + \|x_n - p\|\), further implies the sequence \(\{y_n\}\) is bounded.

Now from (4), we get

\[
(5) \quad \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - 2\alpha_n \Phi(\|x_{n+1} - p\|) + 4L^2 [\Phi^{-1}(a_0)]^2 (3\alpha_n^2 + 2\alpha_n\beta_n). 
\]
Denote
\[ a_n = \|x_n - p\|, \]
\[ b_n = 2\alpha_n, \]
\[ c_n = 4L^2 \left[ \Phi^{-1}(a_0) \right]^2 (3\alpha_n^2 + 2\alpha_n\beta_n). \]
Condition \( \lim_{n \to \infty} \alpha_n = 0 \) ensures the existence of a rank \( n_0 \in \mathbb{N} \) such that \( b_n = 2\alpha_n \leq 1 \), for all \( n \geq n_0 \). Now with the help of \( \sum_{n=0}^{\infty} \alpha_n = \infty \), \( \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n \) and Lemma 2, we obtain from (5) that
\[ \lim_{n \to \infty} \|x_n - p\| = 0, \]
completing the proof.

**Theorem 4.** Let \( E \) be a real Banach space and let \( A : E \to E \) be a Lipschitzian generalized strongly quasi-accretive operator such that \( N(A) \neq \emptyset \). Let \( z \in E \) and \( x_0 \) be an arbitrary initial value in \( E \) for which the steepest descent approximation scheme is defined by
\[ x_{n+1} = x_n - \alpha_n (Ay_n - z), \]
\[ y_n = x_n - \beta_n (Ax_n - z), \quad n = 0, 1, 2 \ldots, \]
where the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) satisfy the following conditions:

(i) \( 0 \leq \alpha_n, \beta_n \leq 1 \),
(ii) \( \sum_{n=0}^{\infty} \alpha_n = +\infty \),
(iii) \( \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n \),
converges strongly to the unique solution of the equation \( Ax = z \).

**Remark 1.** One can easily see that if we take \( \alpha_n = \frac{1}{n^\sigma}; 0 < \sigma < \frac{1}{2} \), then \( \sum \alpha_n = \infty \), but also \( \sum \alpha^2 \notin \infty \). Hence the results of Chidume and Chidume in [4] are not true in general and consequently the results presented in this manuscript are independent of interest.
References


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