Some Dunwoody parameters and cyclic presentations

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Abstract

In this paper, we found the cyclically presented groups obtained from the word $w$ generated with some Dunwoody parameters.

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1 Introduction

Let $F_n$ be the free group on free generators $x_0, x_1, x_2, \ldots, x_{n-1}$. Let $\theta:F_n\rightarrow F_n$ be the automorphism such that

$$\theta(x_i) = x_{i+1}, \ i = 0, 1, \ldots, n - 2, \ \theta(x_{n-1}) = x_0.$$ 

For $w \in F_n$, $G_n(w)$ is defined as $G_n(w) = F_n/R$ where $R$ is the normal closure in $F_n$ of the set

$$\{w, \theta(w), \theta^2(w), \ldots, \theta^{n-1}(w)\} \ [1].$$

For a reduced word $w \in F_n$, the cyclically presented group $G_n(w)$ is given by

$$G_n(w) = \langle x_0, x_1, \ldots, x_{n-1} | w, \theta(w), \theta^2(w), \ldots, \theta^{n-1}(w) \rangle \ [2].$$

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Definition 1. A group $G$ is said to have a cyclic presentation if $G \cong G_n(w)$ for some $n$ and $w$ [3].

Definition 2. A generalized Sieradski group is defined by the cyclic presentation

$$S(r, n) = \langle x_1x_2 \ldots x_n \mid x_ix_{i+2} \ldots x_{i+2r-2} = x_{i+1}x_{i+3} \ldots x_{i+2r-3} >$$

(indices are again modulo $n$) for any two positive integers $r$ and $n \geq 2$. For $r = 2$, these $S(r, n)$ are the Sieradski groups [4].

Let $a, b, c, n$ be integers such that $n > 0$, $a, b, c \geq 0$ and $a + b + c > 0$. Let $\bar{\tau}(a, b, c)$ be the graph shown in Figure 1. This is an infinite graph with an automorphism $\theta$ such that $\theta(u_n) = u_{n+1}$ and $\theta(v_n) = v_{n+1}$. The labels indicate the number of edges joining a pair of vertices. Thus, there are $a$ edges joining $u_1$ and $u_2$. We see that the $\bar{\tau}(a, b, c)$ is $d$-regular where $d = 2a + b + c$. Let $\tau_n = \tau_n(a, b, c)$ denote the graph obtained from $\bar{\tau}(a, b, c)$ by identifying all edges and vertices in each orbit of $\theta^n$. Thus $\tau_n$ has $2n$ vertices [5].

We say that the 6-tuple $(a, b, c, r, s, n)$ has property $M$ if it corresponds to the Heegaard diagram of a 3-manifold. An algorithm determining which 6-tuples have property $M$ is now described. Put $d = 2a + b + c$ and let

$$X = \{-d, -d + 1, \ldots, -1, 1, 2, \ldots, d\}.$$

Let $\alpha, \beta$ be the permutations of $X$ defined as follows:

$$\alpha = (1, d)(2, d-1) \ldots (a, d-a+1)(a+1, -a-c-1)(a+2, -a-c-2) \ldots (a + b, -a - c - b)(a + b + 1, -a - 1)(a + b + 2, -a - 2) \ldots (a + b + c, -a - c)(-1, -d)$$

Figure 1
and
\[ \beta(j) = \begin{cases} 
-(j + r) & \text{if } j > 0 \text{ and } j + r \leq d \text{ or } j < 0 \text{ and } j + r < 0 \\
-(j + r - d) & \text{if } j + r \geq 0
\end{cases} \]

The following theorem characterizes the 6-tuples \((a, b, c, r, s, n)\) that have property \(M\). Detail and the proof of this theorem can be found in [5].

**Theorem 1.1.** Let \(d = 2a + b + c\) be odd. The 6-tuple \((a, b, c, r, s, n)\) has property \(M\) if and only if the following two conditions hold simultaneously:

- \(\alpha \beta\) has two cycles of length \(d\)
- \(ps + q \equiv 0 \pmod{n}\).

Here \(p\) is the difference between the number of arrows pointing down the page and the number of arrows pointing up, whereas \(q\) is the number of arrows pointing from left to right minus the number of arrows pointing from right to left in the oriented path determined by \(\alpha \beta\). The entries in the first cycle of \(\alpha \beta\) contain one vertex from each line segment of the diagram. There exists an integer \(s\) such that \(ps + q \equiv 0 \pmod{n}\). The first cycle of \(\alpha \beta\) and the value of \(s\) can also be used to calculate the word \(w\) of the corresponding cyclic presentation.

## 2 Materials and Methods

We can now state our theorems:

**Theorem 2.1.** The cyclically presented groups obtained from the word \(w\) generated with Dunwoody parameters \((1, b, 0, 2)\) are isomorphic to the groups \(S((d + 1)/2, d)\) when \(b\) is an odd positive integer and \(d = 2a + b + c\).

**Proof.** Suppose for now that \(d > 3\). In this case, there are 2 horizontal arcs and \(b\) diagonal arcs. Thus, the terms in the first cycle of \(\alpha \beta\) for 6-tuple \((1, b, 0, 2, s, n)\) have the following form

\[(1, -2, -4, \ldots, -d + 1, -1, b, b - 2, b - 4, \ldots, 3).\]
According to Figure 2, for the 6-tuple $(1, b, 0, 2, s, n)$, since always $p = -1$ and $q = 1$, from $ps + q \equiv 0 \pmod{n}$, $s$ always takes value 1. Thus, the defining word $w$ corresponding to the 6-tuple $(1, b, 0, 2, s, n)$, calculated using the first cycle of $\alpha \beta$ and the value of $s$, has the following form

\begin{equation}
  x_{d-1}^{-1} x_{d-3}^{-1} x_{d-5}^{-1} \cdots x_{d-b}^{-1} x_0^{-1} x_1 x_3 x_5 \cdots x_b
\end{equation}

For this reduced word $w$, the cyclically presented group $G_d(w)$ is

\begin{align*}
  G_d(w) &= G_d(x_{d-1}^{-1} x_{d-3}^{-1} x_{d-5}^{-1} \cdots x_{d-b}^{-1} x_0^{-1} x_1 x_3 x_5 \cdots x_b) \\
  &= < x_1 x_2, \ldots, x_d | x_1 x_{i+2} \cdots x_{i+d-5} x_{i+d-3} x_{i+d-1} > \\
  &= x_{i+1} x_{i+3} x_{i+5} \cdots x_{i+b} >
\end{align*}

where subscripts are understood to be reduced modulo $d$ to lie in the set \{1, 2, \ldots, d\}. It can be easily seen that the groups $G_d(w)$ have exactly the same presentation as the groups, where

\begin{align*}
  S(r, n) &= < x_1 x_2, \ldots, x_n | x_1 x_{i+2} \cdots x_{i+2r-2} = x_{i+1} x_{i+3} \cdots x_{i+2r-3} > \\
  (indices \ are \ modulo \ n) \ for \ any \ two \ positive \ integers \ r \ and \ n \geq 2, \ given \ by \ Definition \ 2. \ We \ get
\end{align*}

\begin{align*}
  x_{i+d-1} &= x_{i+2r-2}
\end{align*}
from corresponding terms of $G_d(w)$ and $S(r, n)$ and also $i + d - 1 = i + 2r - 2$. Thus, $r = (d + 1)/2$.

Assume now that $d = 3$. For this case, the terms in the first cycle of $\alpha \beta$ of the 4-tuple $(1, 1, 0, 2)$ are

$$(1, -2, -4, \ldots, -d + 1, -1).$$

Notice that when $d = 3$, from (1), the defining word $w$ can be written as $w = x_2^{-1}x_0^{-1}x_1$. For this reduced word $w$, the cyclically presented group $G_3(w)$ is

$$G_3(w) = G_3(x_2^{-1}x_0^{-1}x_1) = \langle x_0, x_1, x_2 | x_i x_{i+2} = x_{i+1}, i \equiv 0 \pmod{3} \rangle.$$

This is $S((d + 1)/2, d)$, so the proof is complete.

**Theorem 2.2.** The cyclically presented group obtained from the word $w$ generated with Dunwoody parameters $(1, b, 0, d - 2)$ has the cyclic presentation

$$\langle x_1 x_2, \ldots, x_d | x_{i+d-1}x_{i+d-3}x_{i+d-5} \cdots x_{i+2}x_i = x_{i+b}x_{i+b-2} \cdots x_{i+5}x_{i+3}x_{i+1} \rangle$$

when $b$ is an odd positive integer and $d = 2a + b + c$.

**Proof.** In this case, there are 2 horizontal arcs and $b$ diagonal arcs. Thus, the terms in the first cycle of $\alpha \beta$ for 6-tuple $(1, b, 0, d - 2, s, n)$ have the following form

$$(1, -b, -b + 2, -b + 4, \ldots, -3, -1, 2, 4, \ldots, d - 1).$$

Figure 3. Heegaard diagram for the 6-tuple $(1, b, 0, d - 2, s, n)$
According to Figure 3, for the 6-tuple \((1,b,0,d-2,s,n)\), since always \(p = 1\) and \(q = 3\), from \(ps + q \equiv 0 \pmod{n}\), \(s\) always takes value -3. Thus, the defining word \(w\) corresponding to the 6-tuple \((1,b,0,d-2,s,n)\), calculated using the first cycle of \(\alpha\beta\) and the value of \(s\), has the following form

\[x_1^{-1}x_3^{-1}x_5^{-1} \cdots x_b^{-1}x_{d-1}x_{d-3}x_{d-5} \cdots x_{d-b}x_0.\]

For this reduced word \(w\), the cyclically presented group \(G_d(w)\) is

\[G_d(w) = G_d(x_1^{-1}x_3^{-1}x_5^{-1} \cdots x_b^{-1}x_{d-1}x_{d-3}x_{d-5} \cdots x_{d-b}x_0) = < x_1x_2, \ldots, x_d | x_{i+d-1}x_{i+d-3}x_{i+d-5} \cdots x_{i+2}x_i = x_{i+b+2}x_{i+5}x_{i+3}x_{i+1}>\]

where all indices are modulo \(d\). This completes the proof.

It is easy to see that the cases \((1,b,0,2)\) and \((1,0,b,2)\), and \((1,b,0,d-2)\) and \((1,0,b,d-2)\), where \(b\) is an odd positive integer, are really the same.

**Theorem 2.3.** The cyclically presented group obtained from the word \(w\) generated with Dunwoody parameters \((a,1,0,a)\) has the cyclic presentation

\[< x_1x_2, \ldots, x_d | x_{i+d-1}x_i = x_{i+d-2}x_{i+d-3}x_{i+d-4} \cdots x_{i+3}x_{i+2}x_{i+1}>\]

when \(a\) is a positive integer and \(d = 2a + b + c\).

**Proof.** In this case, there are \(2a\) horizontal arcs and 1 diagonal arc. Thus, the terms in the first cycle of \(\alpha\beta\) for the 6-tuple \((a,1,0,a,s,n)\) have the following form

\((1,-a,-a+1,3,-a+2,\ldots,-2,d-a-1,-1,d-a)\).
According to Figure 4, for the 6-tuple \((a, 1, 0, a, s, n)\), since always \(p = 1\) and \(q = d\), from \(ps + q \equiv 0 \pmod{n}\), \(s\) always takes value \(-d\). Thus, the defining word \(w\) corresponding to the 6-tuple \((a, 1, 0, a, s, n)\), calculated using the first cycle of \(\alpha \beta\) and the value of \(s\), has the following form

\[x_1^{-1} x_2 x_3^{-1} \cdots x_d^{-1} x_{d-3} x_{d-2} x_{d-1} x_0.\]

For this reduced word \(w\), the cyclically presented group \(G_d(w)\) is

\[G_d(w) = G_d(x_1^{-1} x_2 x_3^{-1} \cdots x_{d-4} x_{d-3} x_{d-2} x_{d-1} x_0) = \langle x_1 x_2, \ldots, x_d \mid x_{i+d-1} x_i = x_{i+d-2} x_{i+d-3} x_{i+d-4} \cdots x_{i+3} x_{i+2} x_{i+1} \rangle\]

where all indices are modulo \(d\). We are done.

**Theorem 2.4.** The cyclically presented group obtained from the word \(w\) generated with Dunwoody parameters \((a, 1, 0, a + 1)\) has the cyclic presentation

\[\langle x_1 x_2, \ldots, x_d \mid x_{i+d-1} x_i x_{i+2} x_{i+3} \cdots x_{i+d-4} x_{i+d-3} x_{i+d-2} = x_i x_{i+d-1} \rangle\]

when \(a\) is a positive integer and \(d = 2a + b + c\).
Proof. In this case, there are $2\,a$ horizontal arcs and 1 diagonal arc. Thus, the terms in the first cycle of $\alpha\beta$ for the 6-tuple $(a, 1, 0, a + 1, s, n)$ have the following form

$$(1, -a - 1, -1, a, -2, a - 1, \ldots, 3, -a + 1, 2, -a).$$

According to Figure 5, for the 6-tuple $(a, 1, 0, a + 1, s, n)$, since always $p = -1$ and $q = d - 2$, from $ps + q \equiv 0 \pmod{n}$, $s$ always takes value $d - 2$. Thus, the defining word $w$ corresponding to the 6-tuple $(a, 1, 0, a + 1, s, n)$, calculated using the first cycle of $\alpha\beta$ and the value of $s$, has the following form

$$x_{d-1}^{-1}x_0^{-1}x_1^{-1}x_2^{-1}x_3 \cdots x_{d-4}^{-1}x_{d-3}^{-1}x_{d-2}.$$ 

For this reduced word $w$, the cyclically presented group $G_d(w)$ is

$$G_d(w) = G_d(x_{d-1}^{-1}x_0^{-1}x_1^{-1}x_2^{-1}x_3 \cdots x_{d-4}^{-1}x_{d-3}^{-1})$$

$$= < x_1x_2, \ldots, x_d | x_i x_{i+1}^{-1}x_{i+2} x_{i+3} \cdots x_{i+d-4}x_{i+d-3}^{-1}x_{i+d-2} = x_i x_{i+d-1} >$$

where all indices are modulo $d$. This completes the proof.

It is easy to see that the cases $(a, 1, 0, a)$ and $(a, 0, 1, a)$, and $(a, 1, 0, a + 1)$ and $(a, 0, 1, a + 1)$, where $a$ is a positive integer, are really the same.
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