On some integral classes of integral operators

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Abstract

Let $A$ be the class of the functions $f$ which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$. The object of the present paper is to derive univalence conditions of certain integral operators for $f(z) \in A$ and $f(z)$ has the form: $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$.

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1 Introduction

Let $A$ be the class of the functions $f(z)$ which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $S$ the class of the functions $f(z) \in A$ which are univalent in $U$.

In this paper we consider the integral operators

\begin{equation}
F_\alpha(z) = \int_0^z [f'(u)]^\alpha \, du
\end{equation}

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\[ H_{\beta,\gamma}(z) = \left\{ \beta \int_{0}^{z} u^{\beta-1} \left[ f'(u) \right]^\gamma \, du \right\}^{\frac{1}{\beta}} \]

\[ L_{\beta}(z) = \left[ \beta \int_{0}^{z} u^{\beta-1} \left[ f''(u) \right] \, du \right]^{\frac{1}{\beta}} \]

2 Preliminary Results

We need the following theorems.

**Lemma 2.1.** [1]. If \( f(z) \in A \) satisfies

\[ (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U \]

then \( f(z) \in S \).

**Theorem 2.2.** [3]. Let \( \alpha \) be a complex number, \( \Re \alpha > 0 \) and \( f(z) \in A \). If

\[ \frac{1 - |z|^{2 \Re \alpha}}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad \text{for all } z \in U, \]

for any complex number \( \beta, \Re \beta \geq \Re \alpha \) the function

\[ F_{\beta}(z) = \left[ \beta \int_{0}^{z} u^{\beta-1} f'(u)du \right]^{\frac{1}{\beta}} \]

is in the class \( S \).

**Theorem 2.3.** [2]. If the function \( g(z) \) is regular in \( U \) and \( |g(z)| < 1 \) in \( U \), then for all \( \xi \in U \) and \( z \in U \) the following inequalities hold:

\[ \left| \frac{g(\xi) - g(z)}{1 - g(z)g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right|, \]

\[ |g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \]
the equalities hold only in the case \( g(z) = \frac{e^{(z+u)}}{1+uz}, \) where \(|\epsilon| = 1\) and \(|u| < 1\).

**Remark 2.4. [2]** For \( z = 0\), from inequality (2.4). We have

\[
(2.6) \quad \left| \frac{g(\xi) - g(0)}{1 - g(0)g(\xi)} \right| \leq |\xi|
\]

and, hence

\[
(2.7) \quad |g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}.
\]

Considering \( g(0) = a \) and \( \xi = z \),

\[
(2.8) \quad |g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}
\]

for all \( z \in U \).

## 3 Main Results

**Theorem 3.1.** Let \( \alpha \) be a complex number and \( f(z) \in A \),

\[
f(z) = z + \sum_{k=3}^{\infty} a_k z^k.
\]

If

\[
(3.1) \quad \left| \frac{f''(z)}{f'(z)} \right| < 1, \quad z \in U
\]

and

\[
(3.2) \quad |\alpha| \leq 4
\]

then the function

\[
(3.3) \quad F_\alpha(z) = \int_0^z \left[ f'(u) \right]^\alpha du
\]

is in the class \( S \).

**Proof.** The function \( F_\alpha(z) \) is regular in \( U \). Let us consider the function

\[
(3.4) \quad p(z) = \frac{1}{|\alpha|} \frac{F''_\alpha(z)}{F'_\alpha(z)}
\]
where the constant $|\alpha|$ satisfies the inequality (3.2).

The function $p(z)$ is regular in $U$. From (3.4) and (3.3) we obtain

$$p(z) = \frac{\alpha}{|\alpha|} \frac{f''(z)}{f'(z)}.$$  \hspace{1cm} (3.5)

Using (3.1) and (3.5) we get

$$|p(z)| \leq 1, \ z \in U$$  \hspace{1cm} (3.6)

and we have $p(0) = 0$.

By Remark 2.4 we have

$$|p(z)| \leq |z|, \ z \in U.$$  \hspace{1cm} (3.7)

From (3.4) and (3.7) we obtain

$$\frac{1}{|\alpha|} \left| \frac{F''_\alpha(z)}{F'_\alpha(z)} \right| \leq |z|, \ z \in U$$  \hspace{1cm} (3.8)

and

$$\left(1 - |z|^2\right) \left| \frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| \leq |\alpha| \max_{|z|<1} \left(1 - |z|^2\right) |z|^2.$$  \hspace{1cm} (3.9)

Because $\max_{|z|<1} \left(1 - |z|^2\right) |z|^2 = \frac{1}{4}$, from (3.9) and (3.2) we get

$$\left(1 - |z|^2\right) \left| \frac{zF''_\alpha(z)}{F'_\alpha(z)} \right| \leq 1, \ z \in U.$$  \hspace{1cm} (3.10)

By Lemma 2.1 it results that the function $F_\alpha(z) \in S$.

**Theorem 3.2.** Let $\gamma$ be a complex number and the function $f(z) \in A$, $f(z) = z + \sum_{k=3}^{\infty} a_k z^k$. If

$$\left| \frac{f''(z)}{f'(z)} \right| < 1, \ z \in U$$  \hspace{1cm} (3.11)
and

\begin{equation}
|\gamma| \leq 4
\end{equation}

then for any complex number \(\beta\), \(\text{Re} \beta \geq 1\) the function

\begin{equation}
H_{\beta, \gamma}(z) = \left\{ \beta \int_0^z u^{\beta-1} [f'(u)]^\gamma \, du \right\}^{\frac{1}{\beta}}
\end{equation}

is in the class \(S\).

**Proof.** Let us consider the function

\begin{equation}
g(z) = \int_0^z [f'(u)]^\gamma \, du.
\end{equation}

The function

\begin{equation}
p(z) = \frac{1}{|\gamma|} \frac{g''(z)}{g'(z)},
\end{equation}

where the constant \(|\gamma|\) satisfies the inequality (3.12), is regular in \(U\).

From (3.15) and (3.14) we obtain

\begin{equation}
p(z) = \frac{\gamma}{|\gamma|} \frac{f''(z)}{f'(z)}.
\end{equation}

and using (3.11) we have

\(|p(z)| \leq 1, \ z \in U\)

Remark 2.4 applied to the function \(p(z)\) give

\begin{equation}
\frac{1}{|\gamma|} \left| \frac{g''(z)}{g'(z)} \right| \leq |z|, \ z \in U
\end{equation}

and, hence

\begin{equation}
(1 - |z|^2) \left| \frac{z g''(z)}{g'(z)} \right| \leq |\gamma| \max_{|z| < 1} (1 - |z|^2) |z|^2.
\end{equation}
From (3.18) and (3.12) we obtain

\[(1 - |z|^2) \left| \frac{zg''(z)}{g'(z)} \right| \leq 1, \quad z \in U.\]

By Theorem 2.2 for \(\text{Re}\ \alpha = 1\), it results that \(H_{\beta,\gamma}(z) \in S\).

**Theorem 3.3.** Let \(\beta\) a complex number, \(\text{Re} \ \beta \geq 1\) and \(f(z) \in A\), 
\(f(z) = z + a_3z^3 + \ldots, \frac{f(z)}{z} \neq 0, \ z \in U.\) If

\[(3.20) \quad \left| \frac{f''(z)}{f'(z)} \right| \leq 4, \quad z \in U\]

then the function

\[(3.21) \quad L_{\beta}(z) = \left[ \beta \int_0^z u^{\beta-1} [f'(u)] \ du \right]^{\frac{1}{\beta}}\]

is in the class \(S\).

**Proof.** Let us consider the function

\[g(z) = \frac{1}{4} \frac{f''(z)}{f'(z)}\]

which is regular in \(U\). Remark 2.4 applied to the function \(g(z)\) give

\[(3.22) \quad \frac{1}{4} \left| \frac{f''(z)}{f'(z)} \right| \leq |z|, \quad z \in U\]

and, hence, we obtain

\[(3.23) \quad (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 4 \max_{|z| < 1} (1 - |z|^2) |z|^2, \quad z \in U\]

Since \(\max_{|z| < 1} (1 - |z|^2) |z|^2 = \frac{1}{4}\), from (3.23) we have

\[(3.24) \quad (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad z \in U.\]

From (3.24) and Theorem 2.2 for \(\text{Re} \ \alpha = 1\), we obtain \(F_{\beta}(z) \in S\).
References


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