A Note on Heredity for Terraced Matrices

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Abstract

A terraced matrix $M$ is a lower triangular infinite matrix with constant row segments. In this paper it is seen that when $M$ is a bounded linear operator on $\ell^2$, hyponormality, compactness, and noncompactness are inherited by the “immediate offspring” of $M$. It is also shown that the Cesàro matrix cannot be the immediate offspring of another hyponormal terraced matrix.

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1 Introduction

Assume that $\{a_n\}$ is a sequence of complex numbers such that the associated terraced matrix $M = \begin{pmatrix} a_0 & 0 & 0 & \ldots \\ a_1 & a_1 & 0 & \ldots \\ a_2 & a_2 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ is a bounded linear operator on

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these matrices have been studied in [2] and [3]. We recall that $M$ is said to be \textit{hyponormal} on $\ell^2$ if $\langle [M^*, M]f, f \rangle = \langle (M^*M - MM^*)f, f \rangle \geq 0$ for all $f$ in $\ell^2$. It seems natural to ask whether hyponormality is inherited by the terraced matrix arising from any subsequence $\{a_{nk}\}$. To see that the answer is no, we consider the case where $M = C = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2} & 0 & \ldots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, the Cesàro matrix. In [4, Corollary 5.1] it is seen that the terraced matrix associated with the subsequence $\{\frac{1}{2n+1} : n = 0, 1, 2, \ldots\}$ is not hyponormal, although the Cesàro matrix itself is known to be a hyponormal operator on $\ell^2$ (see [1]).

Consequently, we turn our attention to a more modest result and consider hereditary properties of the terraced matrix arising from one special subsequence; we will regard $M' = \begin{pmatrix} a_1 & 0 & 0 & \ldots \\ a_2 & a_2 & 0 & \ldots \\ a_3 & a_3 & a_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ as the \textit{immediate offspring} of $M$, for $M'$ is itself the terraced matrix that results from removing the first row and the first column from $M$. Note that $M' = U^*MU$ where $U$ is the unilateral shift.

\section{Main Result}

**Theorem 2.1.** (a) $M'$ inherits from $M$ the property of hyponormality.

(b) $M$ is compact if and only if $M'$ is compact.

**Proof.** (a) We must show that $[(M')^*, M'] \equiv (M')^*M' - M'(M')^* \geq 0$. Critical to the proof is the fact that $(M')^* M' = U^*\{M^*MU\}$, which can be verified by computing that both sides of the equation are equal to the
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reverse-L-shaped matrix

\[
\begin{pmatrix}
0 & 0 & 0 & b_1 & b_2 & b_3 & \ldots \\
0 & 0 & b_3 & b_2 & b_1 & 0 & \ldots \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

where \( b_n = \sum_{k=n}^{\infty} |a_k|^2 \); also, it can be verified that

\[
M' = \begin{pmatrix}
|a_1|^2 & a_1 \overline{a_2} & a_1 \overline{a_3} & \ldots \\
\overline{a_1} a_2 & 2 |a_2|^2 & 2a_2 \overline{a_3} & \ldots \\
\overline{a_1} a_3 & 2 \overline{a_2} a_3 & 3 |a_3|^2 & \ldots \\
& \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
= (U^* M) \{(UU^*) (M^* U) \}
\]

and that

\[
U^*\{(MM^*) U \} = \begin{pmatrix}
2 |a_1|^2 & 2a_1 \overline{a_2} & 3a_1 \overline{a_3} & \ldots \\
\overline{a_1} a_2 & 3 |a_2|^2 & 3a_2 \overline{a_3} & \ldots \\
\overline{a_1} a_3 & 3 \overline{a_2} a_3 & 4 |a_3|^2 & \ldots \\
& \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
= (U^* M) \{I (M^* U) \}.
\]

Consequently, we have

\[
[(M')^*,M'] = (M')^* M' - M' (M')^*
\]

\[
= U^*\{(M^* M) U \} - (U^* M) \{(UU^*) (M^* U) \}
\]

\[
= U^*\{(M^* M) U \} - U^*\{(MM^*) U \} + U^*\{(MM^*) U \} - (U^* M) \{(UU^*) (M^* U) \}
\]

\[
= U^*\{(M^* M) U \} - U^*\{(MM^*) U \} + (U^* M) \{I (M^* U) \} - (U^* M) \{(UU^*) (M^* U) \}
\]

\[
= U^*\{(M^* M) U \} + (M^* U) \{I - (UU^*) (M^* U) \}.
\]

Since \( M \) is hyponormal (by hypothesis) and \( I - UU^* \geq 0 \), we find that

\[
\langle ((M')^*, M') f, f \rangle =
= \langle [M^*,M] U f, U f \rangle + \langle (I - UU^*) (M^* U) f, (M^* U) f \rangle
\geq 0 + 0 = 0
\]

for all \( f \) in \( \ell^2 \).

This completes the proof of part (a).
(b) We prove only one direction. Suppose $M'$ is compact. It follows that $UM'U^*$ is also compact. Note that $M - UM'U^*$ has nonzero entries only in the first column; these entries are precisely the terms of the sequence $\{a_n\}$. Since $M$ is bounded, we must have $\sum_{n=0}^{\infty} |a_n|^2 = \|Me_0\|^2 < \infty$, where $e_0$ belongs to the standard orthonormal basis for $\ell^2$; consequently, $M - UM'U^*$ is a Hilbert-Schmidt operator on $\ell^2$ and is therefore compact. Thus $M = UM'U^* + (M - UM'U^*)$ is compact, since it is the sum of two compact operators.

**Corollary 2.1.** Assume $M''$ is the terraced matrix obtained by removing the first $k$ rows and the first $k$ columns from $M$, for some fixed positive integer $k > 1$. (a) $M''$ inherits from $M$ the property of hyponormality. (b) $M$ is compact if and only if $M''$ is compact.

### 3 Other Results

We note that normality (occurring when $M$ commutes with $M^*$) and quasinormality (occurring when $M$ commutes with $M^*M$) are also inherited properties for terraced matrices, but those turn out to be trivialities. The proofs are left to the reader.

**Theorem 3.1.** (a) If $M$ is normal, then $a_n = 0$ for all $n \geq 1$ and $M' = 0$. (b) If $M$ is quasinormal, then $a_n = 0$ for all $n \geq 1$ and $M' = 0$.

In closing, we consider a question about the most famous terraced matrix, the Cesàro matrix $C$. Is $C$ the immediate offspring of some other hyponormal terraced matrix; that is, does there exist a hyponormal terraced matrix $A$ such that $C = A' = U^*AU$? The matrix $A$ would have to be generated by $\{a_n\}$ with $a_0$ yet to be determined and $a_n = \frac{1}{n}$ for $n \geq 1$. Then $L = \lim_{n \to +\infty} (n+1)a_n = \lim_{n \to +\infty} \frac{n+1}{n} = 1$. From [3, Theorems 2.5 and 2.6] we
conclude that the spectrum is \( \sigma(A) = \{ \lambda : |\lambda - 1| \leq 1 \} \cup \{a_0\} \) and that \( A \) cannot be hyponormal since \( \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} > 1 = L^2 \). Thus we see that nonhyponormality is not inherited by the immediate offspring of a terraced matrix.

References


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