General Mathematics Vol. 15, No. 4 (2007), 149-160

On the Unified Class of functions of Complex Order involving Dziok–Srivastava Operator[1]

T.N. Shanmugam, S. Sivasubramanian, G. Murugusundaramoorthy

Abstract

In the present investigation, we consider an unified class of functions of complex order. We obtain a necessary and sufficient condition for functions in these classes.

2000 Mathematical Subject Classification: 30C45, 30C55, 30C80

Key words: Starlike functions of complex order, convex functions of complex order, subordination

1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions

$$f(z) = z + a_2z^2 + a_3z^2 + \cdots \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is subordinate to an univalent function $g \in \mathcal{A}$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$. Let Ω be the family of analytic functions $\omega(z)$ in the

[1] *Received 15 July, 2007*

Accepted for publication (in revised form) 20 September, 2007

unit disc Δ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(\omega(z))$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. The class $S^*(\phi)$, introduced and studied by Ma and Minda [10], consists of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta).$$

The functions $h_{\phi n}$ $(n = 2, 3, \ldots)$ by

$$\frac{zh'_{\phi n}(z)}{h_{\phi n}(z)} = \phi(z^{n-1}), \quad h_{\phi n}(0) = 0 = h'_{\phi n}(0) - 1.$$

We write $h_{\phi 2}$ simply as h_ϕ. The functions $h_{\phi n}$ are all functions in $S^*(\phi)$.

Recently, Ravichandran et al. [14] defined classes related to the class of starlike functions of complex order defined as

Definition 1.1. *Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [10], Shanmugam and Sivasubramanian [19] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\frac{\alpha z^2 f''(z) + zf'(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \phi(z),$$

where $\phi(z)$ satisfies the condition mentioned in Definition 1.1.

For any two analytic functions $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum\limits_{n=0}^{\infty} b_n z^n$, the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f*g)(z)$ is defined by

$$(f*g)(z)=\sum_{n=0}^{\infty} a_n b_n z^n.$$

For complex parameters $\alpha_1,\alpha_2,...,\alpha_q$ and $\beta_1,\beta_2,...,\beta_s$ with $(\beta_j\neq 0,-1,-2,...;j=1,2,...,s)$, we define the *generalized hypergeometric function* ${}_qF_s(z)$ by

$$(1.2)\quad {}_qF_s(\alpha_1,\alpha_2,...,\alpha_q;\beta_1,\beta_2,...,\beta_s;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n...(\alpha_q)_n}{(\beta_1)_n(\beta_2)_n...(\beta_s)_n(1)_n}\quad z^n$$

$$(\ q\leq s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\,;z\in\mathcal{U})$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(1.3)\qquad (\lambda)_n=\begin{cases}1 & \text{for } n=0\\ \lambda\ (\lambda+1)...(\lambda+n-1) & \text{for } n=1,2,3...\end{cases}.$$

Corresponding to a function $h_p(\alpha_1,\alpha_2,...\alpha_q;\beta_1,\beta_2,...\beta_s;z)$ defined by

$$h(\alpha_1,\alpha_2,...\alpha_q;\beta_1,\beta_2,...\beta_s;z)=z\,{}_qF_s(\alpha_1,\alpha_2,...,\alpha_q;\beta_1,\beta_2,...,\beta_s;z),$$

we consider the Dziok–Srivastava operator [3]

$$H(\alpha_1,\alpha_2,...,\alpha_q;\beta_1,\beta_2,...,\beta_s)f(z):\mathcal{A}\longrightarrow\mathcal{A},$$

defined by the convolution

$$H(\alpha_1,\alpha_2,...\alpha_q;\beta_1,\beta_2,...\beta_s)f(z)=h(\alpha_1,\alpha_2,...\alpha_q;\beta_1,\beta_2,...\beta_s;z)*f(z).$$

We observe that, for a function f of the form (1.1), we have

$$(1.4)\qquad H(\alpha_1,\alpha_2,...,\alpha_q;\beta_1,\beta_2,...,\beta_s)f(z)=z+\sum_{n=k}^{\infty}\Gamma_n a_n z^n$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}, ..., (\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}, ..., (\beta_s)_{n-1}(1)_{n-1}}. \tag{1.5}$$

For convenience, we write

$$H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) := H_{q,s}(\alpha_1) \tag{1.6}$$

Thus, through a simple calculations, we obtain

$$z\left(H_{q,s}(\alpha_1)f(z)\right)' = \alpha_1 H_{q,s}(\alpha_1 + 1)f(z) - (\alpha_1 - 1)H_{q,s}(\alpha_1)f(z). \tag{1.7}$$

The Dziok–Srivastava operator $H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s)$ includes various other linear operators which were considered in earlier works in the literature. For $s = 1$ and $q = 2$, we obtain the linear operator:

$$\mathcal{F}(\alpha_1, \alpha_2; \beta_1)f(z) = H(\alpha_1, \alpha_2; \beta_1)f(z),$$

which was introduced by Hohlov [6]. Moreover, putting $\alpha_2 = 1$, we obtain the Carlson-Shaffer operator [1]:

$$\mathcal{L}(\alpha_1, \beta_1)f(z) = H(\alpha_1, 1; \beta_1)f(z).$$

Ruscheweyh [16] introduced an operator

$$\mathcal{D}^m f(z) = \frac{z}{(1-z)^m} * f(z) \quad (m \geq -1; f \in \mathcal{A}). \tag{1.8}$$

From the equation (1.7), we have

$$\mathcal{D}^\lambda f(z) = H(\lambda + 1, 1; 1)f(z). \tag{1.9}$$

In this, we introduce a more general class of complex order $M_{q,s,b,\alpha}(\phi) = M_{\alpha_1,...,\alpha_q,\beta_1,...,\beta_s,b,\alpha}(\phi)$ which we define below.

Definition 1.2. *Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps* *the unit disk* Δ *onto a region starlike with respect to* 1 *which is symmetric with respect to the real axis. Then the class* $M_{q,s,b,\alpha}(\phi)$ *consists of all analytic functions* $f \in \mathcal{A}$ *satisfying*

$$1 + \frac{1}{b}\left(\Psi(q, s, z) - 1\right) \prec \phi(z), \quad (\alpha \geq 0).$$

where
$\Psi_{q,s}(\alpha_1)f(z) := \Psi(\alpha_1....\alpha_q; \beta_1, ..., \beta_s; z)f :=$
(1.10)

$$\frac{\alpha(\alpha_1+1)H(\alpha_1+2)f(z) + (1-2\alpha_1\alpha)H(\alpha_1+1)f(z) - (1-\alpha)(\alpha_1-1)H(\alpha_1)f(z)f(z)}{(1-\alpha)H(\alpha_1)f(z)f(z) + \alpha H(\alpha_1+1)f(z)}.$$

We also denote,
(i) For $q = 2$ and $s = 1$, $M_{q,s,b,\alpha}(\phi) \equiv F(b, \alpha)(\phi)$.
(ii) For $q = 2$, $s = 1$ and $\alpha_2 = 1$, $M_{q,s,b,\alpha}(\phi) \equiv M(\alpha_1, \beta_1, b, \alpha)(\phi)$.
(iii) For $q = 2$, $s = 1$, $\alpha_1 = 1 + m$, $\alpha_2 = 1$ and $\beta_1 = 1$, $M_{q,s,b,\alpha}(\phi) \equiv M(m, b, \alpha)(\phi)$.

Clearly, for $q = s = 1$, $\alpha_1 = \beta_1 = 1$,

$$M_{1,1,b,0}(\phi) \equiv S^*_b(\phi) \quad \text{and} \quad M_{1,1,b,1}(\phi) \equiv C_b(\phi).$$

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $M_{q,s,b,\alpha}(\phi)$ which we have defined. The motivation of this paper is to generalize the results obtained by Ravichandran et al. [14] and also Srivastava and Lashin [20].

Our results includes several known results. To see this,let $M_{1,1,b,1}(A, B) \equiv S^*(A, B, b)$ and $M_{1,1,b,1}(A, B) \equiv C(A, B, b)$ ($b \neq 0$, complex) denote the classes $S^*_b(\phi)$ and $C_b(\phi)$ respectively when

$$\phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).$$

The class $S^*(A, B, b)$ and therefore the class $S^*_b(\phi)$ specialize to several well-known classes of univalent functions for suitable choices of A, B and b. The class $S^*(A, B, 1)$ is denoted by $S^*(A, B)$. Some of these classes are listed below where $ST(b)$ denotes $1 + \frac{1}{b}(\frac{zf'(z)}{f(z)} - 1)$.

1. $S^*(1,-1,1)$ is the class S^* of starlike functions [5, 2, 13].

2. $S^*(1,-1,b)$ is the class of starlike functions of complex order introduced by Wiatrowski [21]. We denote this class by S_b^*.

3. $S^*(1,-1,1-\beta)\,,\ 0\leq\beta<1,$ is the class $S^*(\beta)$ of starlike functions of order β. This class was introduced by Robertson [15].

4. $S^*(1,0,b)$ is the set defined by $|ST(b)-1|<1$.

5. $S^*(\beta,0,b)$ is the set defined by $|ST(b)-1|<\beta$, $0\leq\beta<1$.

6. $S^*(\beta,-\beta,b)$ is the set defined by $\left|\frac{ST(b)-1}{ST(b)+1}\right|<\beta$, $0\leq\beta<1$.

To prove our main result, we need the following results.

The following result follows a result of Ruscheweyh [16] for functions in the class $S^*(\phi)$ (see Ruscheweyh [17, Theorem 2.37, pages 86–88]).

Lemma 1.1. *Let ϕ be a convex function defined on Δ, $\phi(0)=1$. Define $F(z)$ by*

$$F(z)=z\exp\left(\int_0^z\frac{\phi(x)-1}{x}dx\right). \tag{1.11}$$

Let $q(z)=1+c_1z+\cdots$ be analytic in Δ. Then

$$1+\frac{zq'(z)}{q(z)}\prec\phi(z) \tag{1.12}$$

if and only if for all $|s|\leq1$ and $|t|\leq1$, we have

$$\frac{q(tz)}{q(sz)}\prec\frac{sF(tz)}{tF(sz)}. \tag{1.13}$$

Lemma 1.2. *[11, Corollary 3.4h.1, p.135] Let $q(z)$ be univalent in Δ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)/\varphi(q(z))$ is starlike, then*

$$zp'(z)\varphi(p(z))\prec zq'(z)\varphi(q(z)),$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

2 Main Results

By making use of Lemma 1.1, we have the following:

Theorem 2.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in M_{q,s,b,\alpha}(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s\left[((1-\alpha)H_{q,s}(\alpha_1)f(tz)+\alpha H_{q,s}(\alpha_1+1)f(tz)\right]}{t\left[(1-\alpha)H_{q,s}(\alpha_1)f(sz)+\alpha H_{q,s}(\alpha_1+1)f(sz)\right]}\right)^{1/b} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.1}$$

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{(1-\alpha)H_{q,s}(\alpha_1)f(z)+\alpha H_{q,s}(\alpha_1+1)f(z)}{z}\right)^{1/b}. \tag{2.2}$$

By taking logarithmic derivative of $p(z)$ given by (2.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left\{\frac{(1-\alpha)z(H_{q,s}(\alpha_1)f(z))'+\alpha z(H_{q,s}(\alpha_1+1)f(z))'}{(1-\alpha)H_{q,s}(\alpha_1)f(z)+\alpha H_{q,s}(\alpha_1+1)f(z)} - 1\right\}. \tag{2.3}$$

By using the identity (1.7), we obtain by a straight forward computation, we get,

$$1+\frac{zp'(z)}{p(z)} = 1+\frac{1}{b}\left(\Psi_{q,s}(\alpha_1)f(z)-1\right)$$

where

$$\Psi_{q,s}(\alpha_1)f(z) = \frac{\alpha(\alpha_1+1)H(\alpha_1+2)f(z)+(1-2\alpha_1\alpha)H(\alpha_1+1)f(z)-(1-\alpha)(\alpha_1-1)H(\alpha_1)f(z)f(z)}{(1-\alpha)H(\alpha_1)f(z)f(z)+\alpha H(\alpha_1+1)f(z)}. \tag{2.4}$$

The result now follows from Lemma 1.1.

For $q = 2$ and $s = 1$, in Theorem 2.1, we get the following result in terms of the Hohlov operator.

Corollary 2.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in F_{b,\alpha}(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s\left[((1-\alpha)F(\alpha_1,\alpha_2;\beta_1)f(tz)+\alpha F(\alpha_1+1,\alpha_2;\beta_1)f(tz)\right]}{t\left[(1-\alpha)F(\alpha_1,\alpha_2;\beta_1)f(sz)+\alpha F(\alpha_1+1,\alpha_2;\beta_1)f(sz)\right]}\right)^{1/b} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.5}$$

For $q = 2$, $s = 1$ and $\alpha_2 = 1$, in Theorem 2.1, we get the following result in terms of the Carlson–Shaffer operator.

Corollary 2.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in M_{\alpha_1,\beta_1,b,\alpha}(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$\left(\frac{s\left[((1-\alpha)L(\alpha_1;\beta_1)f(tz)+\alpha L(\alpha_1+1;\beta_1)f(tz)\right]}{t\left[(1-\alpha)L(\alpha_1;\beta_1)f(sz)+\alpha L(\alpha_1+1;\beta_1)f(sz)\right]}\right)^{1/b} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.6}$$

For $q = 2$, $s = 1$, $\alpha_1 = 1 + m$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.1, we get the following result in terms of the Ruscheweyh derivative.

Corollary 2.3. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in M_{m,b,\alpha}(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$\left(\frac{s\left[(1-\alpha)D^m f(tz)+\alpha D^{m+1}f(tz)\right]}{t\left[(1-\alpha)D^m f(sz)+\alpha D^{m+1}f(sz)\right]}\right)^{1/b} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.7}$$

For $q = s = 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha = 0$ in Theorem 2.1, we get

Corollary 2.4. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in S_b^*(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$\left(\frac{sf(tz)}{tf(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.8}$$

For $q = s = 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha = 1$ in Theorem 2.1, we get

Corollary 2.5. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in C_b(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$\left(\frac{f'(tz)}{f'(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}.$$

As an immediate consequence of the above Corollary 2.4, we have

Corollary 2.6. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. If $f \in S_b^*(\phi)$, then we have*

$$\frac{f(z)}{z} \prec \left(\frac{F(z)}{z}\right)^b. \tag{2.9}$$

Theorem 2.2. *Let ϕ starlike with respect to 1 and $F(z)$ is given by (1.11) be starlike. If $f \in M_{q,s,b,\alpha}(\phi)$, then we have*

$$\frac{(1-\alpha)H_{q,s}(\alpha_1)f(z)+\alpha H_{q,s}(\alpha_1+1)f(z)}{z} \prec \left(\frac{F(z)}{z}\right)^b. \tag{2.10}$$

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \left(\frac{(1-\alpha)H_{q,s}(\alpha_1)f(z)+\alpha H_{q,s}(\alpha_1+1)f(z)}{z}\right)^{1/b}, \quad q(z) := \left(\frac{F(z)}{z}\right).$$

Then a computation yields

$$1+\frac{zp'(z)}{p(z)} = 1+\frac{1}{b}\left(\Psi(z)-1\right)$$

where $\Psi_{q,s}(\alpha_1)f(z)$ is as defined in (2.4) and

$$\frac{zq'(z)}{q(z)} = \left(\frac{zF'(z)}{F(z)}-1\right) = \phi(z)-1.$$

Since $f \in M^*_{b,\alpha}(\phi)$, we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left(\Psi(a,c,z)-1\right) \prec \phi(z)-1 = \frac{zq'(z)}{q(z)}.$$

The result now follows by an application of Lemma 1.2.

By taking $\phi(z) = (1+z)/(1-z)$, $q = s = 1$, $\alpha_1 = \beta_1 = 1$ and $\alpha = 0$ in Theorem 2.2, we get the following result of Srivastava and Lashin [20]:

Example 2.1. *If $f \in S^*_b$, then*

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}.$$

By taking $\phi(z) = (1+z)/(1-z)$, $q = s = 1$, $\alpha_1 = \beta_1 = 1$ and $\alpha = 1$ in Theorem 2.2, we get another result of Srivastava and Lashin [20]:

Example 2.2. *If $f \in C_b$, where $C_b = C_b(\phi)$ when $\phi(z) = \frac{1+z}{1-z}$ then*

$$f'(z) \prec \frac{1}{(1-z)^{2b}}.$$

References

[1] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., 15(1984), 737–745.

[2] P. L. Duren, *Univalent functions*, Springer, New York, 1983.

[3] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. 103 (1999), no. 1, 1–13.

[4] A. Gangadharan, V. Ravichandran and T. N. Shanmugam, *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl. 211 (1) (1997), 301–313.

[5] A. W. Goodman, *Univalent functions* Vol. I, II, Mariner, Tampa, FL, 1983.

[6] Ju. E. Hohlov and *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat. , no. 10(197), 83–89,(1978).

[7] I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. (2) 3 (1971), 469–474.

[8] W. Janowski, *Extremal problem for a family of functions with positive real part and for some related families*, Ann. Polon. Math. 23(1970), 159–177.

[9] R. J. Libera, *Some radius of convexity problem*, Duke Math. J. 31 (1964), 143–157.

[10] W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Internat. Press, Cambridge, MA.

[11] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Dekker, New York, 2000.

[12] M. A. Nasr and M. K. Aouf,*Starlike function of complex order*, J. Natur. Sci. Math. 25 (1) (1985), 1–12.

[13] Ch. R. Pommerenke, *Univalent functions*, Vandenhoeck, ruprecht in Göttingen, 1975.

[14] V. Ravichandran, Yasar Polatoglu, Metin Bolcal and Arsu Sen, *Certain Subclasses of Starlike and Convex Functions of Complex Order,* (To appear)

[15] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. 37 (1936), 374–408.

[16] St. Ruscheweyh,*A subordination theorem for* Φ*-like functions*, J. London Math. Soc. 13 (1976), 275–280.

[17] S. Ruscheweyh, *Convolutions in geometric function theory*, Presses Univ. Montréal, Montreal, Que., 1982.

[18] L. Špaček, *Contribution à la théorie des functions univalents*, Časopis Pěst. Mat 62 (1932) 12–19.

[19] T.N. Shanmugam, and S. Sivasubramanian,*On the Fekete-Szegö problem for some subclasses of analytic functions*, J. Inequal. Pure Appl. Math.,6(3), (2005), Article 71, 6 pp. [ONLINE: http://jipam.vu.edu.au/article.php?sid=544]

[20] H. M. Srivastava, and A. Y. Lashin,*Some applications of the Briot-Bouquet differential subordination*, J. Inequal. Pure Appl. Math. 6 (2) (2005), Article 41, 7 pp. (electronic).

[21] P. Wiatrowski, *The coefficients of a certain family of holomorphic functions*, Zeszyty Nauk. Uniw. Łódz. Nauki Mat. Przyrod. Ser. II No. 39 Mat. (1971), 75–85.

T.N. Shanmugam
Department of Mathematics
College of Engineering
Anna University, Chennai-600 025
Tamilnadu, India
E-mail: shan@annauniv.edu

S. Sivasubramanian
Department of Mathematics
Easwari Engineering College
Ramapuram, Chennai-600 08
Tamilnadu, India
E-mail: sivasaisastha@rediffmail.com

G. Murugusundaramoorthy
Department of Mathematics,
Vellore Institute of Technology,
Deemed University ,
Vellore-632 014, India
E-mail: gmsmoorthy@yahoo.com