On the Unified Class of functions of Complex Order involving Dziok–Srivastava Operator

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Abstract

In the present investigation, we consider an unified class of functions of complex order. We obtain a necessary and sufficient condition for functions in these classes.

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1 Introduction

Let \( \mathcal{A} \) be the class of all analytic functions

\[
    f(z) = z + a_2 z^2 + a_3 z^3 + \cdots
\]

in the open unit disk \( \Delta = \{z \in \mathbb{C} : |z| < 1\} \). A function \( f \in \mathcal{A} \) is subordinate to an univalent function \( g \in \mathcal{A} \), written \( f(z) \prec g(z) \), if \( f(0) = g(0) \) and \( f(\Delta) \subseteq g(\Delta) \). Let \( \Omega \) be the family of analytic functions \( \omega(z) \) in the...
unit disc $\Delta$ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(\omega(z))$.

Let $S$ be the subclass of $A$ consisting of univalent functions. The class $S^*(\phi)$, introduced and studied by Ma and Minda [10], consists of functions in $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta).$$

The functions $h_{\phi_n} (n = 2, 3, \ldots)$ by

$$\frac{zh'_{\phi_n}(z)}{h_{\phi_n}(z)} = \phi(z^{n-1}), \quad h_{\phi_n}(0) = h'_{\phi_n}(0) - 1.$$ 

We write $h_{\phi^2}$ simply as $h_{\phi}$. The functions $h_{\phi_n}$ are all functions in $S^*(\phi)$.

Recently, Ravichandran et al. [14] defined classes related to the class of starlike functions of complex order defined as

**Definition 1.1.** Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S^*_b(\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [10], Shanmugam and Sivasubramanian [19] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\frac{\alpha z^2 f''(z) + zf'(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} \prec \phi(z),$$
where $\phi(z)$ satisfies the condition mentioned in Definition 1.1.

For any two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$ 

For complex parameters $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ with $(\beta_j \neq 0, -1, -2, ..., j = 1, 2, ..., s)$, we define the generalized hypergeometric function $qF_s(z)$ by

$$qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n ... (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n ... (\beta_s)_n (1)_n} z^n \quad (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U})$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{for } n = 0 \\ \lambda (\lambda + 1) ... (\lambda + n - 1) & \text{for } n = 1, 2, 3, ... \end{cases}.$$

Corresponding to a function $h_p(\alpha_1, \alpha_2, ... \alpha_q; \beta_1, \beta_2, ... \beta_s; z)$ defined by

$$h(\alpha_1, \alpha_2, ... \alpha_q; \beta_1, \beta_2, ... \beta_s; z) = z qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z),$$

we consider the Dziok–Srivastava operator $H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) f(z) : \mathcal{A} \rightarrow \mathcal{A}$, defined by the convolution

$$H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) f(z) = h(\alpha_1, \alpha_2, ... \alpha_q; \beta_1, \beta_2, ... \beta_s; z) * f(z).$$

We observe that, for a function $f$ of the form (1.1), we have

$$H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s) f(z) = z + \sum_{n=k}^{\infty} \Gamma_n a_n z^n$$
where

\[ \Gamma_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}, \ldots, (\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}, \ldots, (\beta_s)_{n-1}(1)_{n-1}}. \]

For convenience, we write

\[ H(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s) := H_{q,s}(\alpha_1) \]

Thus, through a simple calculations, we obtain

\[ z (H_{q,s}(\alpha_1)f(z))' = \alpha_1 H_{q,s}(\alpha_1 + 1)f(z) - (\alpha_1 - 1)H_{q,s}(\alpha_1)f(z). \]

The Dziok–Srivastava operator \( H(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s) \) includes various other linear operators which were considered in earlier works in the literature. For \( s = 1 \) and \( q = 2 \), we obtain the linear operator:

\[ \mathcal{F}(\alpha_1, \alpha_2; \beta_1)f(z) = H(\alpha_1, \alpha_2; \beta_1)f(z), \]

which was introduced by Hohlov [6]. Moreover, putting \( \alpha_2 = 1 \), we obtain the Carlson-Shaffer operator [1]:

\[ \mathcal{L}(\alpha_1, \beta_1)f(z) = H(\alpha_1, 1; \beta_1)f(z). \]

Ruscheweyh [16] introduced an operator

\[ D^m f(z) = \frac{z}{(1 - z)^m} * f(z) \quad (m \geq -1; f \in A). \]

From the equation (1.7), we have

\[ D^\lambda f(z) = H(\lambda + 1, 1; 1)f(z). \]

In this, we introduce a more general class of complex order

\[ M_{q,s,b,\alpha}(\phi) = M_{\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s, b, \alpha}(\phi) \]

which we define below.

**Definition 1.2.** Let \( b \neq 0 \) be a complex number. Let \( \phi(z) \) be an analytic function with positive real part on \( \Delta \) with \( \phi(0) = 1, \phi'(0) > 0 \) which maps
the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M_{q,s,b,\alpha}(\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b} (\Psi(q,s,z) - 1) < \phi(z), \quad (\alpha \geq 0).$$

where

$$\Psi_{q,s}(\alpha_1)f(z) := \Psi(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)f :=$$

$$\frac{\alpha(\alpha_1 + 1)H(\alpha_1 + 2)f(z) + (1 - 2\alpha_1\alpha)H(\alpha_1 + 1)f(z) - (1 - \alpha)(\alpha_1 - 1)H(\alpha_1)f(z)f(z)}{(1 - \alpha)H(\alpha_1)f(z)f(z) + \alpha H(\alpha_1 + 1)f(z)}.$$

We also denote,

(i) For $q = 2$ and $s = 1$, $M_{q,s,b,\alpha}(\phi) \equiv F(b, \alpha)(\phi)$.

(ii) For $q = 2$, $s = 1$ and $\alpha_2 = 1$, $M_{q,s,b,\alpha}(\phi) \equiv M(\alpha_1, \beta_1, b, \alpha)(\phi)$.

(iii) For $q = 2$, $s = 1$, $\alpha_1 = 1 + m$, $\alpha_2 = 1$ and $\beta_1 = 1$, $M_{q,s,b,\alpha}(\phi) \equiv M(m, b, \alpha)(\phi)$.

Clearly, for $q = s = 1$, $\alpha_1 = \beta_1 = 1$,

$$M_{1,1,b,0}(\phi) \equiv S^*_b(\phi) \quad \text{and} \quad M_{1,1,b,1}(\phi) \equiv C_b(\phi).$$

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $M_{q,s,b,\alpha}(\phi)$ which we have defined. The motivation of this paper is to generalize the results obtained by Ravichandran et al. [14] and also Srivastava and Lashin [20].

Our results includes several known results. To see this, let $M_{1,1,b,1}(A, B) \equiv S^*(A, B, b)$ and $M_{1,1,b,1}(A, B) \equiv C(A, B, b)$ ($b \neq 0$, complex) denote the classes $S^*_b(\phi)$ and $C_b(\phi)$ respectively when

$$\phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).$$

The class $S^*(A, B, b)$ and therefore the class $S^*_b(\phi)$ specialize to several well-known classes of univalent functions for suitable choices of $A$, $B$ and $b$. The class $S^*(A, B, 1)$ is denoted by $S^*(A, B)$. Some of these classes are listed below where $ST(b)$ denotes $1 + \frac{1}{b}(\frac{zf(z)}{f(z)} - 1)$.
1. $S^{*}(1,-1,1)$ is the class $S^{*}$ of starlike functions $[5,2,13]$.

2. $S^{*}(1,-1,b)$ is the class of starlike functions of complex order introduced by Wiatrowski [21]. We denote this class by $S_{b}^{*}$.

3. $S^{*}(1,-1,1-\beta), 0 \leq \beta < 1,$ is the class $S^{*}(\beta)$ of starlike functions of order $\beta$. This class was introduced by Robertson [15].

4. $S^{*}(1,0,b)$ is the set defined by $|ST(b) - 1| < 1$.

5. $S^{*}(\beta,0,b)$ is the set defined by $|ST(b) - 1| < \beta, 0 \leq \beta < 1$.

6. $S^{*}(\beta,-\beta,b)$ is the set defined by $\left| \frac{ST(b) - 1}{ST(b) + 1} \right| < \beta, 0 \leq \beta < 1$.

To prove our main result, we need the following results.

The following result follows a result of Ruscheweyh [16] for functions in the class $S^{*}(\phi)$ (see Ruscheweyh [17] Theorem 2.37, pages 86–88).

**Lemma 1.1.** Let $\phi$ be a convex function defined on $\Delta$, $\phi(0) = 1$. Define $F(z)$ by

\begin{equation}
F(z) = z \exp \left( \int_{0}^{z} \frac{\phi(x) - 1}{x} \, dx \right).
\end{equation}

Let $q(z) = 1 + c_{1}z + \cdots$ be analytic in $\Delta$. Then

\begin{equation}
1 + \frac{zq'(z)}{q(z)} \prec \phi(z)
\end{equation}

if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

\begin{equation}
\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}.
\end{equation}

**Lemma 1.2.** [11] Corollary 3.4h.1, p.135] Let $q(z)$ be univalent in $\Delta$ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)/\varphi(q(z))$ is starlike, then

\[ zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)), \]

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.
2 Main Results

By making use of Lemma 1.1, we have the following:

**Theorem 2.1.** Let \( \phi(z) \) and \( F(z) \) be as in Lemma 1.1. The function \( f \in M_{q,s,b,\alpha}(\phi) \) if and only if for all \( |s| \leq 1 \) and \( |t| \leq 1 \), we have

\[
\left( \frac{s}{t} \left[ (1-\alpha)H_{q,s}(\alpha_1)f(tz) + \alpha H_{q,s}(\alpha_1+1)f(sz) \right] \right)^{1/b} \preceq \frac{sF(tz)}{tF(sz)}.
\]

**Proof:** Define the function \( p(z) \) by

\[
p(z) := \left( \frac{(1-\alpha)H_{q,s}(\alpha_1)f(z) + \alpha H_{q,s}(\alpha_1+1)f(z)}{z} \right)^{1/b}.
\]

By taking logarithmic derivative of \( p(z) \) given by (2.2), we get

\[
\frac{zp'(z)}{p(z)} = \frac{1}{b} \left\{ \frac{(1-\alpha)z(H_{q,s}(\alpha_1)f(z))' + \alpha z(H_{q,s}(\alpha_1+1)f(z))'}{(1-\alpha)H_{q,s}(\alpha_1)f(z) + \alpha H_{q,s}(\alpha_1+1)f(z)} - 1 \right\}.
\]

By using the identity (1.7), we obtain by a straightforward computation, we get,

\[
1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b} (\Psi_{q,s}(\alpha_1)f(z) - 1)
\]

where

\[
\Psi_{q,s}(\alpha_1)f(z) = \frac{(1-\alpha)(\alpha_1+1)f(z) + (1-2\alpha_1\alpha)H_{q,s}(\alpha_1)f(z) - (1-\alpha)(\alpha_1-1)H_{q,s}(\alpha_1)f(z)}{(1-\alpha)H_{q,s}(\alpha_1)f(z) + \alpha H_{q,s}(\alpha_1+1)f(z)}.
\]

The result now follows from Lemma 1.1.

For \( q = 2 \) and \( s = 1 \), in Theorem 2.1 we get the following result in terms of the Hohlov operator.

**Corollary 2.1.** Let \( \phi(z) \) and \( F(z) \) be as in Lemma 1.1. The function \( f \in F_{b,\alpha}(\phi) \) if and only if for all \( |s| \leq 1 \) and \( |t| \leq 1 \), we have

\[
\left( \frac{s}{t} \left[ (1-\alpha)F(\alpha_1,\alpha_2;\beta_1)f(tz) + \alpha F(\alpha_1+1,\alpha_2;\beta_1)f(tz) \right] \right)^{1/b} \preceq \frac{sF(tz)}{tF(sz)}.
\]
For $q = 2$, $s = 1$ and $\alpha_2 = 1$, in Theorem 2.1 we get the following result in terms of the Carlson–Shaffer operator.

**Corollary 2.2.** Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in M_{\alpha_1, \beta_1, b, \alpha} (\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$ f((1 - \alpha)L(\alpha_1; \beta_1)f(tz) + \alpha L(\alpha_1 + 1; \beta_1)f(tz)) \leq \frac{sF(tz)}{tF(sz)}.$$  

For $q = 2$, $s = 1$, $\alpha_1 = 1 + m$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.1 we get the following result in terms of the Ruscheweyh derivative.

**Corollary 2.3.** Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in M_{m, b, \alpha} (\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$ f((1 - \alpha)D^{m}f(tz) + \alpha D^{m+1}f(tz)) \leq \frac{sF(tz)}{tF(sz)}.$$  

For $q = s = 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha = 0$ in Theorem 2.1 we get

**Corollary 2.4.** Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in S_{b}^{*} (\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$ sf(tz) \leq \frac{sF(tz)}{tF(sz)}.$$  

For $q = s = 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha = 1$ in Theorem 2.1 we get

**Corollary 2.5.** Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in C_{b} (\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$ \left( \frac{f'(tz)}{f'(sz)} \right)^{\frac{1}{\delta}} \leq \frac{sF(tz)}{tF(sz)}.$$  

As an immediate consequence of the above Corollary 2.4, we have

**Corollary 2.6.** Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. If $f \in S_{b}^{*} (\phi)$, then we have

$$ \frac{f(z)}{z} \leq \left( \frac{F(z)}{z} \right)^{\delta}.$$
Theorem 2.2. Let \( \phi \) starlike with respect to 1 and \( F(z) \) is given by \((1.11)\) be starlike. If \( f \in M_{q,s,b,\alpha}(\phi) \), then we have

\[
(2.10) \quad \left( 1 - \alpha \right) H_{q,s}(\alpha_1)f(z) + \alpha H_{q,s}(\alpha_1 + 1)f(z) \prec \left( \frac{F(z)}{z} \right)^b.
\]

**Proof.** Define the functions \( p(z) \) and \( q(z) \) by

\[
p(z) := \left( \frac{(1 - \alpha)H_{q,s}(\alpha_1)f(z) + \alpha H_{q,s}(\alpha_1 + 1)f(z)}{z} \right)^{1/b}, \quad q(z) := \left( \frac{F(z)}{z} \right).
\]

Then a computation yields

\[
1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}(\Psi(z) - 1)
\]

where \( \Psi_{q,s}(\alpha_1)f(z) \) is as defined in \((2.4)\) and

\[
\frac{zq'(z)}{q(z)} = \left( \frac{zF'(z)}{F(z)} - 1 \right) = \phi(z) - 1.
\]

Since \( f \in M_{b,\alpha}^*(\phi) \), we have

\[
\frac{zp'(z)}{p(z)} = \frac{1}{b}(\Psi(a,c,z) - 1) \prec \phi(z) - 1 = \frac{zq'(z)}{q(z)}.
\]

The result now follows by an application of Lemma 1.2.

By taking \( \phi(z) = (1 + z)/(1 - z) \), \( q = s = 1 \), \( \alpha_1 = \beta_1 = 1 \) and \( \alpha = 0 \) in Theorem 2.2 we get the following result of Srivastava and Lashin [20]:

**Example 2.1.** If \( f \in S_b^* \), then

\[
\frac{f(z)}{z} \prec \frac{1}{(1 - z)^{2b}}.
\]

By taking \( \phi(z) = (1 + z)/(1 - z) \), \( q = s = 1 \), \( \alpha_1 = \beta_1 = 1 \) and \( \alpha = 1 \) in Theorem 2.2 we get another result of Srivastava and Lashin [20]:

**Example 2.2.** If \( f \in C_b \), where \( C_b = C_b(\phi) \) when \( \phi(z) = \frac{1 + z}{1 - z} \) then

\[
f'(z) \prec \frac{1}{(1 - z)^{2b}}.
\]
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