A criteria of \( \phi \)-like functions\(^1\)

Sushma Gupta, Sukjit Singh and Sukhwinden Singh

Abstract

In this paper, we obtain some sufficient conditions for a normalized analytic function to be \( \phi \)-like and starlike of order \( \alpha \).

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1 Introduction

Let \( \mathcal{A} \) be the class of functions \( f \) which are analytic in the unit disc \( E = \{ z : |z| < 1 \} \) and are normalized by the conditions \( f(0) = f'(0) - 1 = 0 \).

Denote by \( S^*(\alpha) \) and \( K(\alpha) \), the classes of starlike functions of order \( \alpha \) and convex functions of order \( \alpha \) respectively, which are analytically defined as follows

\[
S^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in E \right\}
\]

and

\[
K(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in E \right\}
\]

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where \( \alpha \) is a real number such that \( 0 \leq \alpha < 1 \). We shall use \( S^* \) and \( K \) to denote \( S^*(0) \) and \( K(0) \), respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

Let \( f \) and \( g \) be analytic in \( E \). We say that \( f \) is subordinate to \( g \) in \( E \), written as \( f(z) \prec g(z) \) in \( E \), if \( g \) is univalent in \( E \), \( f(0) = g(0) \) and \( f(E) \subset g(E) \).

Denote by \( S^*[A,B] \), \(-1 \leq B < A \leq 1 \), the class of functions \( f \in A \) which satisfy

\[
\frac{zf'(z)}{f(z)} < \frac{1+Az}{1+Bz}, \quad z \in E.
\]

Note that \( S^*[1-2\alpha,-1] = S^*(\alpha), \quad 0 \leq \alpha < 1 \) and \( S^*[1,-1] = S^* \).

A function \( f, f'(0) \neq 0 \), is said to be close-to-convex in \( E \), if and only if, there is a starlike function \( h \) (not necessarily normalized) such that

\[
\Re \frac{zf'(z)}{h(z)} > 0, \quad z \in E.
\]

Let \( \phi \) be analytic in a domain containing \( f(E) \), \( \phi(0) = 0 \) and \( \Re \phi'(0) > 0 \), then, the function \( f \in A \) is said to be \( \phi \)-like in \( E \) if

\[
\Re \frac{zf'(z)}{\phi(f(z))} > 0, \quad z \in E.
\]

This concept was introduced by L. Brickman [1]. He proved that an analytic function \( f \in A \) is univalent if and only if \( f \) is \( \phi \)-like for some \( \phi \). Later, Ruscheweyh [8] investigated the following general class of \( \phi \)-like functions:

Let \( \phi \) be analytic in a domain containing \( f(E) \), \( \phi(0) = 0, \phi'(0) = 1 \) and \( \phi(w) \neq 0 \) for \( w \in f(E) - \{0\} \), then the function \( f \in A \) is called \( \phi \)-like with respect to a univalent function \( q, q(0) = 1 \), if

\[
\frac{zf'(z)}{\phi(f(z))} < q(z), \quad z \in E.
\]

In the present note, we obtain some sufficient conditions for a normalized analytic function to be \( \phi \)-like. In [9], Silverman defined the class \( G_b \) as

\[
G_b = \left\{ f \in A : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, \quad z \in E \right\}
\]
and proved that the functions in the class $G_b$ are starlike in $E$. Later on, this class was studied extensively by Tuneski [4,11,12,13,14,15]. As particular cases, we obtain many interesting results for the class $G_b$. Most of the results proved by Tuneski follow as corollaries to our theorem.

2 Preliminaries

We shall need following definition and lemmas to prove our results.

**Definition 2.1.** A function $L(z, t), z \in E$ and $t \geq 0$ is said to be a subordination chain if $L(., t)$ is analytic and univalent in $E$ for all $t \geq 0$, $L(z, .)$ is continuously differentiable on $[0, \infty)$ for all $z \in E$ and $L(z, t_1) \prec L(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

**Lemma 2.1** [5, page 159]. The function $L(z, t) : E \times [0, \infty) \to \mathbb{C}$, ($\mathbb{C}$ is the set of complex numbers), of the form $L(z, t) = a_1(t)z + \ldots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t \to \infty} |a_1(t)| = \infty$, is said to be a subordination chain if and only if $\Re \left[ \frac{\partial L}{\partial z} \frac{\partial L}{\partial t} \right] > 0$ for all $z \in E$ and $t \geq 0$.

**Lemma 2.2** [3]. Let $F$ be analytic in $E$ and let $G$ be analytic and univalent in $E$ except for points $\zeta_0$ such that $\lim_{z \to \zeta_0} F(z) = \infty$, with $F(0) = G(0)$. If $F \not\preceq G$ in $E$, then there is a point $z_0 \in E$ and $\zeta_0 \in \partial E$ (boundary of $E$) such that $F(|z| < |z_0|) \subset G(E)$, $F(z_0) = G(\zeta_0)$ and $z_0 F'(z_0) = mz_0 G'(\zeta_0)$ for some $m \geq 1$.

3 Main Result

**Lemma 3.1.** Let $\gamma, \Re \gamma \geq 0$, be a complex number. Let $q$ be univalent function such that either $\frac{zq'(z)}{q(z)}$ is starlike in $E$ or $\frac{1}{q(z)}$ is convex in $E$. If an analytic function $p$, satisfies the differential subordination

\begin{equation}
(3.1) \quad 1 - \frac{\gamma}{p(z)} + \frac{zp'(z)}{p^2(z)} < 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)}, \quad p(0) = q(0) = 1, \quad z \in E,
\end{equation}
then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

**Proof.** Let us define a function

$$h(z) = 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)}, \ z \in E. \quad (3.2)$$

Firstly, we will prove that $h(z)$ is univalent in $E$ so that the subordination (3.1) is well-defined in $E$. Differentiating (3.2) and simplifying a little, we get

$$zh'(z) = \gamma + zQ'(z)Q(z), \ z \in E,$$

where $Q(z) = \frac{zq'(z)}{q^2(z)}$. In view of the given conditions, we obtain

$$\Re \frac{zh'(z)}{Q(z)} > 0, \ z \in E.$$

Thus, $h(z)$ is close-to-convex and hence univalent in $E$. We need to show that $p \prec q$. Suppose to the contrary that $p \not\prec q$ in $E$. Then by Lemma 2.2, there exist points $z_0 \in E$ and $\zeta_0 \in \partial E$ such that $p(z_0) = q(\zeta_0)$ and $z_0p'(z_0) = m\zeta q'(\zeta_0), \ m \geq 1$. Then

$$1 - \frac{\gamma}{p(z_0)} + \frac{z_0p'(z_0)}{p^2(z_0)} = 1 - \frac{\gamma}{q(\zeta_0)} + \frac{m\zeta q'(\zeta_0)}{q^2(\zeta_0)}, \ z \in E. \quad (3.3)$$

Consider a function

$$L(z, t) = 1 - \frac{\gamma}{q(z)} + (1 + t) \frac{zq'(z)}{q^2(z)}, \ z \in E. \quad (3.4)$$

The function $L(z, t)$ is analytic in $E$ for all $t \geq 0$ and is continuously differentiable on $[0, \infty)$ for all $z \in E$. Now,

$$a_1(t) = \left( \frac{\partial L(z, t)}{\partial z} \right)_{(0,t)} = q'(0)(\gamma + 1 + t).$$

In view of the condition that $\Re \gamma \geq 0$, we get $|\arg(\gamma + 1 + t)| \leq \pi/2$. Also, as $q$ is univalent in $E$, so, $q'(0) \neq 0$. Therefore, it follows that $a_1(t) \neq 0$ and
\[ \lim_{t \to \infty} |a_1(t)| = \infty. \] A simple calculation yields
\[ \frac{z \frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}} = \gamma + (1 + t) \frac{z Q'(z)}{Q(z)}, \quad z \in E. \]

Clearly
\[ \Re \frac{z \frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}} > 0, \quad z \in E, \]
in view of given conditions. Hence, \( L(z, t) \) is a subordination chain. Therefore, \( L(z, t_1) \prec L(z, t_2) \) for \( 0 \leq t_1 \leq t_2 \). From (3.4), we have \( L(\zeta_0, t) \notin h(E) \) for \( |\zeta_0| = 1 \) and \( t \geq 0 \). In view of (3.3) and (3.4), we can write
\[ 1 - \frac{\gamma}{p(z_0)} + \frac{z_0 p'(z_0)}{p^2(z_0)} = L(\zeta_0, m - 1) \notin h(E), \]
where \( z_0 \in E, |\zeta_0| = 1 \) and \( m \geq 1 \) which is a contradiction to (3.1). Hence, \( p \prec q \). This completes the proof of the Lemma.

**Theorem 3.1.** Let \( \gamma, \Re \gamma \geq 0 \), be a complex number. Let \( q, q(0) = 1 \), be a univalent function such that \( \frac{z q'(z)}{q(z)} \) is starlike in \( E \) or, equivalently, \( \frac{1}{q(z)} \) is convex in \( E \). If an analytic function \( f \in A \) satisfies the differential subordination
\[ 1 + \frac{1 - \gamma + z f''(z)/f'(z)}{zf'(z)/f(z)}/(z f'(z)/\phi(f(z)))' < 1 - \frac{\gamma}{q(z)} + \frac{z q'(z)}{q^2(z)}, \quad z \in E, \]
for some function \( \phi \), analytic in a domain containing \( f(E) \), \( \phi(0) = 0, \phi'(0) = 1 \) and \( \phi(w) \neq 0 \) for \( w \in f(E) - \{0\} \), then \( \frac{z f'(z)}{\phi(f(z))} \prec q(z) \) and \( q(z) \) is the best dominant.

**Proof.** The proof of the theorem follows by writing \( p(z) = \frac{z f'(z)}{\phi(f(z))} \) in Lemma 3.1.

In particular, for \( \phi(w) = w \) and \( q(z) = \frac{z q'(z)}{q(z)} \) in Theorem 3.1, we obtain the following result.

**Theorem 3.2.** Let \( \gamma, \Re \gamma \geq 0 \), be a complex number. Let \( g \in A \) be such that \( \frac{z g'(z)}{g(z)} = q(z) \) is univalent in \( E \). Assume that either \( \frac{z q'(z)}{q(z)} \) is starlike
in $E$ or $\frac{1}{q(z)}$ is convex in $E$. If an analytic function $f \in A$ satisfies the differential subordination

$$1 - \gamma + zf''(z)/f'(z) < 1 - \gamma + zg''(z)/g'(z), \quad z \in E,$$

then $\frac{zf''(z)}{f'(z)} < \frac{zg''(z)}{g'(z)}$.

### 4 Applications to univalent functions

In this section, we obtain a criterion for a normalized analytic function to be $\phi$-like. As an application of Theorems 3.1 and 3.2, we obtain some new conditions and also few existing conditions for a function to be in the class $S^*$ and $S^*(\alpha)$.

When the dominant is $q(z) = \frac{1+Az+Bz}{1+Bz}$. We observe that $q$ is univalent in $E$ and $\frac{1}{q(z)}$ is convex in $E$ where $-1 \leq B < A \leq 1$. From Theorem 3.1, we deduce the following result.

**Theorem 4.1.** Let $\gamma, \Re \gamma \geq 0$, be a complex number and $A$ and $B$ be real numbers $-1 \leq B < A \leq 1$. Let $f \in A$ satisfy the differential subordination

$$1 + \frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} - \frac{(\phi(f(z)))'}{f'(z)} < 1 - \gamma \frac{1 + Bz}{1 + A} + \frac{(A - B)z}{(1 + A)z^2}, \quad z \in E,$$

for some function $\phi$, analytic in a domain containing $f(E)$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(E) - \{0\}$, then $\frac{zf'(z)}{\phi(f(z))} < \frac{1+Az}{1+Bz}, \quad z \in E$.

As an example, if we take $\gamma = i, A = 0, B = -1$ in Theorem 4.1, we obtain the following result.

**Example 4.1.** Let $f \in A$ satisfy

$$\left| \frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} - \frac{(\phi(f(z)))'}{f'(z)} + i \right| < \sqrt{2}, \quad z \in E,$$

then $\frac{zf'(z)}{\phi(f(z))} < \frac{1}{1-\gamma}, \quad z \in E$. 


In particular, for $\gamma = 0$ and $A = 1, B = -1$, Theorem 4.1, reduces to the following result.

**Corollary 4.1.** Let $f \in A$ satisfy the differential subordination

$$1 + zf''(z)/f'(z) < 2z/(1 + z)^2, \quad z \in E,$$

for some function $\phi$, analytic in a domain containing $f(E)$, $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(E) - \{0\}$, then $\Re \frac{zf'(z)}{\phi(f(z))} > 0, \quad z \in E$.

Note that several such results are available for different substitutions of constants $A, B$.

For the dominant $q(z) = \frac{1 + Az}{1 + Bz}$, Theorem 3.2 gives us the following result.

**Theorem 4.2.** Let $\gamma, \Re \gamma \geq 0$, be a complex number and $A$ and $B$ be real numbers $-1 \leq B < A \leq 1$. Let $f \in A$ satisfy the differential subordination

$$1 - \gamma + zf''(z)/f'(z) \prec 1 + Bz + (A - B)z/(1 + Az)^2, \quad z \in E,$$

then $f \in S^*[A, B]$.

Writing $\gamma = 1$ in Theorem 4.2, we obtain the following result.

**Corollary 4.2.** If $f \in A$ satisfies the differential subordination

$$f''(z)f(z)/f^2(z) \prec 1 - \gamma + Bz + (A - B)z/(1 + Az)^2, \quad z \in E, \quad -1 \leq B < A \leq 1,$$

then $f \in S^*[A, B]$.

Writing $A = 0$ in Theorem 4.2, we obtain the following result.

**Corollary 4.3.** Let $f \in A$ satisfy

$$\left| 1 - \gamma + zf''(z)/f'(z) \right| < (1 + \gamma)B, \quad z \in E, \quad \gamma \geq 0, \quad 0 < B \leq 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1 + Bz}, \quad z \in E.$$
In particular, for $\gamma = 1$, in Corollary 4.3, we obtain the following result.

**Corollary 4.4.** Let $f \in A$ satisfy

$$\left| \frac{f(z)f''(z)}{f'^2(z)} \right| < 2B, \ z \in E, \ 0 < B \leq 1,$$

then

$$\frac{zf'(z)}{f(z)} < \frac{1}{1 + Bz}, \ z \in E.$$

The selection of $B = 0$ in Theorem 4.2 gives us the following result.

**Corollary 4.5.** Let $f \in A$ satisfy

$$\frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 - \frac{\gamma}{1 + Az} + \frac{Az}{(1 + Az)^2}, \ z \in E, \ \gamma \geq 0, \ 0 < A \leq 1,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < A, \ z \in E.$$

In particular, for $\gamma = 0$ in Corollary 4.5, we obtain the following result.

**Corollary 4.6.** Let $f \in A$ satisfy

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 + \frac{Az}{(1 + Az)^2}, \ z \in E, \ 0 < A \leq 1,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < A, \ z \in E.$$

Taking $\gamma = 1$ in corollary 4.5, we obtain the following result.

**Corollary 4.7.** If

$$\frac{f(z)f''(z)}{f'^2(z)} < 1 - \frac{1}{(1 + Az)^2}, \ z \in E, \ 0 < A \leq 1,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < A, \ z \in E.$$

**Remark 4.1.** (i) Writing $\gamma = 0$ in Theorem 4.2, we obtain the Theorem 2.3 in [14].
(ii) Writing $A = -1, B = 1$ in Theorem 4.2, we obtain Theorem 1 of [15].

(iii) Taking $A = 1, B = -1, \gamma = 0$ in Theorem 4.2, we obtain Theorem 3 in [4].

(iv) Taking $A = -1, B = 1, \gamma = 1$ in Theorem 4.2, we get Theorem 1 in [12].

(v) Taking $A = 0, \gamma = 0$ in Theorem 4.2, we obtain Theorem 1 in [4].

(vi) Writing $A = 0, B = -1, \gamma = 1$ in Theorem 4.2, we obtain the following result:

If $f \in \mathcal{A}$ satisfies $\frac{f''(z)f(z)}{f'^2(z)} < 2z$, $z \in E$, then $f \in S^*(1/2)$.

This is an improvement of Corollary 2 proved in [12].

(vii) Taking $A = -(1 - 2\alpha), B = 1, 0 \leq \alpha < 1$ in Theorem 4.2, we get Theorem 3 in [15].

(viii) Writing $A = -(1 - 2\alpha), B = 1, 0 \leq \alpha < 1$ and $\gamma = 0$ in Theorem 4.2, we obtain Corollary 4(i) in [15].

(ix) Writing $A = -(1 - 2\alpha), B = 1, 0 \leq \alpha < 1$ and for $\gamma = 1$ in Theorem 4.2, Corollary 4(ii) in [15] follows.

(x) For $B = \frac{1-\beta}{\beta}, 1/2 \leq \beta < 1$ in Corollary 4.4, we obtain the result of Robertson [7].

(xi) Taking $q(z) = \frac{2\alpha}{1+z}$ in Theorem 3.2, we obtain Theorem 2 in [15].

References


Sushma Gupta and Sukhjit Singh
Department of Mathematics
S.L.I.E.T., Longowal-148 106 (Punjab) India
E-mail: sushmagupta1@yahoo.com
sukhjit_d@yahoo.com

Sukhwinder Singh
Department of Applied Sciences
B.B.S.B. Engineering College
Fatehgarh Sahib-140 407 (Punjab) India
E-mail: ss_billing@yahoo.co.in