Multivalued Sakaguchi functions

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Abstract

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form \( f(z) = z + a_2 z^2 + \cdots \) which are analytic in the open unit disc \( \mathbb{U} = \{ z \in \mathbb{C} | |z| < 1 \} \). In 1959 [5], K. Sakaguchi has considered the subclass of \( \mathcal{A} \) consisting of those \( f(z) \) which satisfy \( \text{Re} \left( \frac{zf'(z)}{f(z)-f(-z)} \right) > 0 \), where \( z \in \mathbb{U} \). We call such a functions “Sakaguchi Functions”. Various authors have investigated this class ([4], [5], [6]). Now we consider the class of functions of the form \( f(z) = z^\alpha (z + a_2 z^2 + \cdots + a_n z^n + \cdots) \) \((0 < \alpha < 1)\), that are analytic and multivalued in \( \mathbb{U} \), we denote the class of these functions by \( \mathcal{A}_\alpha \), and we consider the subclass of \( \mathcal{A}_\alpha \) consisting of those \( f(z) \) which satisfy \( \text{Re} \left( \frac{zD_\alpha f(z)}{D_\alpha f(z)-D_\alpha f(-z)} \right) > 0 \) \((z \in \mathbb{U})\), where \( D_\alpha f(z) \) is the fractional derivative of order \( \alpha \) of \( f(z) \). We call such a functions “Multivalued Sakaguchi Functions” and denote the class of those functions by \( \mathcal{S}_\alpha \).

The aim of this paper is to investigate some properties of the class \( \mathcal{S}_\alpha \).

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1 Introduction

Let $\mathcal{A}_\alpha$ denote the class of functions $f(z)$ of the form

$$f(z) = z^\alpha \left( z + \sum_{n=2}^{\infty} a_n z^n \right) \quad (0 < \alpha < 1),$$

that are analytic in the open unit disc $U = \{ z \in \mathbb{C} | |z| < 1 \}$. Let $\Omega$ be the class of analytic functions $w(z)$ in $U$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$. Also, denote by $\mathcal{P}$ the class of functions $p(z)$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in $U$ and satisfy $\Re p(z) > 0$ for every $z \in U$.

For analytic functions $g(z)$ in $U$, we recall here the fractional calculus (fractional integrals and fractional derivatives) given by Owa [3], also by Srivastava and Owa [7].

**Definition 1.** The fractional integral of order $\lambda$ for an analytic function $g(z)$ in $U$ is defined by

$$D_z^{-\lambda} g(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{g(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0),$$

where the multiplicity of $(z-\zeta)^{1-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

**Definition 2.** The fractional derivative of order $\lambda$ for an analytic function $g(z)$ in $U$ is defined by

$$D_z^\lambda g(z) = \frac{d}{dz} (D_z^{-1} g(z)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{g(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. 
Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order \((n + \lambda)\) for an analytic function \(g(z)\) in \(U\) is defined by
\[
D_z^{\lambda+n}g(z) = \frac{d^n}{dz^n}(D_z^\lambda g(z)) \quad (0 \leq \lambda < 1, n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}).
\]

Remark 1. From the definitions of the fractional calculus, we see that
\[
D_z^{-\lambda}z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)}z^{k+\lambda} \quad (\lambda > 0, k > 0),
\]
\[
D_z^{\lambda}z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}z^{k-\lambda} \quad (0 \leq \lambda < 1, k > 0),
\]
\[
D_z^{n+\lambda}z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)}z^{k-n-\lambda} \quad (0 \leq \lambda < 1, k > 0, n \in \mathbb{N}_0, k-n \neq -1, -2, \cdots).
\]

Therefore we say that for any real \(\lambda\)
\[
D_z^{\lambda}z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)}z^{k-\lambda} \quad (k > 0, k - \lambda \neq -1, -2, \cdots).
\]

Applying the fractional calculus, we introduce the subclass of \(A_\alpha\).

Definition 4. A function \(f \in A_\alpha\) is said to be Sakaguchi function if \(f(z)\) satisfies
\[
\text{Re}\left(\frac{zD_z^{\alpha}f(z)}{D_z^{2}f(z) - D_z^{2}f(-z)}\right) = p(z) \quad (z \in U)
\]
for some \(p(z) \in \mathcal{P}\). The subclass of \(A_\alpha\) consisting of such functions is denoted by \(S^{\alpha}_s\).

Further, for analytic functions \(h(z)\) and \(s(z)\) in \(U\), \(h(z)\) is said to be subordinate to \(s(z)\) if there exists \(w(z) \in \Omega\) such that \(h(z) = s(w(z))\) \((z \in U)\). We denote this subordination by \(h(z) \prec s(z)\). In particular, if \(s(z)\) is univalent in \(U\), then the subordination \(h(z) \prec s(z)\) is equivalent to \(h(0) = s(0)\) and \(h(U) \subset s(U)\) (see [1]).
2 Main Results

To consider some properties for the class $S^\alpha_s$, we need the following lemma by Jack [2].

Lemma 1. Let $w(z)$ be a non-constant and analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_1 \in U$, then we have

$$z_1 w'(z_1) = kw(z_1),$$

where $k$ is real and $k \geq 1$.

Definition 5. Let us call any transformation which reduces a multivalued function to a single valued a filter for this function.

Lemma 2. Let $\alpha$ be a real number such that $0 < \alpha < 1$, and let

$$f(z) = z^\alpha + \left(z + \sum_{n=2}^{\infty} a_n z^n\right)$$

be an analytic and multivalued function in the open unit disc $U$. Then the $\alpha$–fractional derivative $D_\alpha^\alpha z$ is a filter $f$. Moreover, this filter regularizes $f$.

Property 1. Using the rule for the fractional calculus of the power function $z^\alpha$ and the linear property of the fractional derivatives, we get after simple calculations

$$D_\alpha^\alpha f(z) = D_\alpha^\alpha \left(z^{\alpha+1} + a_2 z^{\alpha+2} + \cdots + a_n z^{\alpha+n} + \cdots\right)$$

$$= \frac{\Gamma(\alpha + 2)}{\Gamma(2)} z + a_2 \frac{\Gamma(\alpha + 3)}{\Gamma(3)} z^2 + \cdots + a_n \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} z^n + \cdots$$

$$= b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots.$$
The inequality (4) shows that $D_\alpha^zf(z)$ is regular and analytic in $\mathbb{U}$.

Conversely, consider the fractional differential equation

$$(5) \quad D_\alpha^zf(z) = s(z) \quad (0 < \alpha < 1).$$

Let us first take the initial condition $f(0) = 0$. Assume that the function $s(z)$ can be expanded in a Taylor series converging for $|z| < 1$, i.e.,

$$(6) \quad s(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^n \quad (z \in \mathbb{U}).$$

Using the rule for the fractional calculus of the power function $z^\alpha$ we write

$$(7) \quad D_\lambda^\alpha z^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - \lambda)} z^{\alpha - \lambda} \quad (0 < \alpha < 1).$$

Taking into account the formula (7) we can look for a solution of the equation (5) in the form of the following power series

$$(8) \quad f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} z^{\alpha + n} \quad (0 < \alpha < 1).$$

Substituting (8) and (6) into the equation (5) and using (7) we get

$$(9) \quad \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} z^n = s(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^n.$$

Comparing the coefficients of the both series in (9), we get

$$(10) \quad a_n = \frac{s^{(n)}(0)}{n!} \frac{\Gamma(n + 1)}{\Gamma(\alpha + n + 1)} = \frac{s^{(n)}(0)}{\Gamma(\alpha + n + 1)}.$$

Therefore under the above assumption, the solution of the equation (5) is

$$f(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{\Gamma(\alpha + n + 1)} z^{\alpha + n}.$$
On the other hand, since the solution \( f(z) \) satisfies the assumed initial condition, we can directly apply \( \alpha \)-th order fractional integration to both sides of the equation \( D_{z}^{\alpha}f(z) = s(z) \), and an application of the composition law for the fractional derivative gives

\[
\begin{align*}
    f(z) &= \sum_{n=0}^{\infty} s^{(n)}(0) \frac{\Gamma(\alpha + n + 1)}{n!} z^{\alpha + n} = \sum_{n=0}^{\infty} s^{(n)}(0) \frac{n!}{\Gamma(\alpha + n + 1)} z^{\alpha + n} \\
    &= \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} \frac{\Gamma(n + 1)}{\Gamma(\alpha + n + 1)} z^{\alpha + n} = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} D_{z}^{-\alpha} z^{n} \\
    &= D_{z}^{-\alpha} \left( \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^{n} \right) = D_{z}^{-\alpha} s(z).
\end{align*}
\]

Therefore we have

\[
D_{z}^{\alpha} f(z) = s(z) \iff f(z) = D_{z}^{-\alpha} s(z).
\]

**Theorem 1.** If \( f \in \mathcal{S}_{\alpha}^{\alpha} \) then the odd starlike function

\[
F(z) = D_{z}^{\alpha} f(z) - D_{z}^{\alpha} f(-z) = 2 \left( \frac{\Gamma(\alpha + 2)}{\Gamma(2)} z + \sum_{k=2}^{\infty} \frac{\Gamma(\alpha + 2k)}{\Gamma(2k)} a_{2k-1} z^{2k-1} \right)
\]

satisfies

\[
\frac{zD_{z}^{\alpha+1} f(z)}{D_{z}^{\alpha} f(z) - D_{z}^{\alpha} f(-z)} + \frac{zD_{z}^{\alpha+1} f(-z)}{D_{z}^{\alpha} f(z) - D_{z}^{\alpha} f(-z)} - 1 < \frac{2z^2}{1 - z^2} = F_{1}(z)
\]

and this result is sharp because the extremal function is the solution of the fractional differential equation

\[
D_{z}^{\alpha} f(z) - D_{z}^{\alpha} f(-z) = \frac{2z}{1 - z^2}.
\]

**Property 2.** We define the function

\[
\frac{D_{z}^{\alpha} f(z) - D_{z}^{\alpha} f(-z)}{2\Gamma(\alpha + 2)z} = (1 - w(z))^{-2} \quad (z \in \mathbb{U}, w(z) \neq 1),
\]
then $w(z)$ is analytic in $U$, $w(0) = 0$ and

$$zF'(z) = zD^{\alpha+1}_z f(z) - D^{\alpha}_z f(-z) + \frac{D^{\alpha}_z f(z) - D^{\alpha}_z f(-z)}{1 - w(z)} = 2zw'(z)$$

Now, it is easy to realize that the subordination (13) is equivalent to $|w(z)| < 1$ for all $z \in U$. Indeed, assume the contrary: then, there exists a $z_1 \in U$, such that $|w(z_1)| = 1$. Then, by Lemma 1, $z_1w'(z_1) = kw(z_1)$ for some real $k \geq 1$. For such $z_1$ we have (form (14))

$$z_1F'(z_1) = \frac{z_1D^{\alpha+1}_z f(z_1) - D^{\alpha}_z f(-z_1)}{D^{\alpha}_z f(z_1) - D^{\alpha}_z f(-z_1)} - 1$$

$$= 2kw(z_1) = F_1(w(z_1)) \notin F_1(U),$$

because $|w(z_1)| = 1$ and $k \geq 1$. But this contradicts (13), so the assumption is wrong, i.e., $|w(z)| < 1$ for every $z \in U$.

The sharpness of this result follows from the fact that

$$F(z) = D^{\alpha}_z f(z) - D^{\alpha}_z f(-z) = \frac{2z}{1 - z^2} \Rightarrow$$

$$zF'(z) = \frac{zD^{\alpha+1}_z f(z) - D^{\alpha}_z f(-z)}{D^{\alpha}_z f(z) - D^{\alpha}_z f(-z)} = \frac{2z^2}{1 - z^2}$$

Corollary 1. If $f(z) \in S^\alpha$, then

$$\left| \left( \frac{2\Gamma(\alpha + 2)z}{D^{\alpha}_z f(z) - D^{\alpha}_z f(-z)} \right)^{\frac{1}{2}} - 1 \right| < 1.$$ 

This inequality is the Marx-Strohhacker inequality for the class $S^\alpha$.

Property 3. This corollary is a simple consequence of Theorem 1.

Corollary 2. If $f(z) \in S^\alpha$, then

$$\frac{\Gamma(\alpha + 2)r}{2(1 + r^2)} \leq |D^{\alpha}_z f(z) - D^{\alpha}_z f(-z)| \leq \frac{\Gamma(\alpha + 2)r}{2(1 - r^2)}.$$
Propertie 4. If $F(z)$ is an odd starlike function, then \[ \frac{r}{1+r^3} \leq |F(z)| \leq \frac{r}{1-r^3}, \]
for $|z| = r$, so by Theorem 1 we obtain (18). This result is sharp because the extremal function is the solution of the fractional differential equation is given (14).

Corollary 3. If $f(z) \in S_{\alpha}^s$, then
\[ (19) \quad \frac{\Gamma(\alpha+2)(1-r)}{(1+r^2)(1+r)} \leq |D_\alpha^s f(z)| \leq \frac{\Gamma(\alpha+2)(1+r)}{(1-r^2)(1-r)}, \]
for $|z| = r$.

Propertie 5. By the definition of the class $S_{\alpha}^s$ and Caratheodory functions we have
\[ (20) \quad \frac{z D_\alpha^s f(z)}{D_\alpha^s f(z) - D_\alpha^s f(-z)} = p(z) \iff z D_\alpha^s f(z) = D_\alpha^s f(z) - D_\alpha^s f(-z) \]
for some $p(z) \in \mathcal{P}$. On the other hand, the well known Caratheodory’s inequality [1]
\[ (21) \quad \frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r}, \]

Together with (18), (20) and (21) yields (19) after simple calculations.

References


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