General Mathematics Vol. 15, Nr. 2–3(2007), 164-176

Subordinations and integral means inequalities

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Abstract

Applying the subordination theorem of J. E. Littlewood [1], and Lemma of S. S. Miller and P. T. Monanu [2] to certain analytic functions, we show an integral means inequality. Further, we obtain an integral means inequality for the first derivative.

2000 Mathematical Subject Classification: Primary 30C45

Key words and phrases: Integral means inequality, analytic function, subordination, first derivative.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $g(z)$ denote the analytic function in $\mathbb{U}$ defined by

$$g(z) = \frac{z}{1-z}. \tag{2}$$

In this paper, we discuss the integral means inequalities of $f(z)$ in $\mathcal{A}$ and $g(z)$ of the form (2), and $f'(z)(f(x) \in \mathcal{A})$ and $g'(z)$. Moreover we show an estimate of $f'(z)$.

We recall the concept of subordination between analytic functions. Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

We need the following subordination theorem of J. E. Littlewood.

Lemma A(Littlewood [1]) *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leqq \int_0^{2\pi} |g(z)|^\mu \, d\theta.$$

Applying the lemma of Littlewood above, H. Silverman [5] showed the integral means inequalities for univalent functions with negative coefficients. S. Owa and T. Sekine [3] proved integral means inequalities with coefficients inequalities for normalized analytic functions and polynomials(see also Sekine et al. [4]).

In addition we need the following Lemma of S. S. Miller and P. T. Mocanu.

Lemma B(Miller and Mocanu [2]) *Let* $g(z) = g_n z^n + g_{n+1} z^{n+1} + \cdots$ *be analytic in* $\mathbb{U}$ *with* $g(z) \neq 0$ *and* $n \geqq 1$. *If* $z_0 = r_0 e^{i\theta_0}\,(r_0 < 1)$ *and*

$$|g(z_0)| = \max_{|z| \leqq |z_0|} |g(z)|$$

then

(i) $\dfrac{z_0 g'(z_0)}{g(z_0)} = k$

and

(ii) $\operatorname{Re}\left(\dfrac{z_0 g''(z_0)}{g'(z_0)}\right) + 1 \geqq k,$

where $k \geqq n \geqq 1$.

2 Integral means for $f(z)$ and $g(z)$

Theorem 1. *Let* $f(z)$ *be in* $\mathcal{A}$ *and* $g(z)$ *be the analytic function given by* (2). *If the function* $f(z)$ *satisfies*

(3)
$$\operatorname{Re}\left\{\alpha f(z) + \beta z f'(z) - \gamma \frac{z f'(z)}{f(z)} - \delta \frac{z f''(z)}{f'(z)}\right\} > -\frac{2\alpha + \beta + 2\gamma - 4\delta}{4} \quad (z \in \mathbb{U})$$

for $\alpha \in \mathbb{R}$, $\beta \geqq 0$, $\beta + 2\gamma \geqq 0$ *and* $\delta \geqq 0$, *then, for* $\mu > 0$ *and* $z = re^{i\theta}\,(0 < r < 1)$

(4)
$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\mu} d\theta.$$

Proof. By applying Lemma A, it suffices to show that

$$f(z) \prec \frac{z}{1 - z}.$$

Let us define the function $w(z)$ by

$$f(z) = \frac{w(z)}{1 - w(z)} \quad (w(z) \neq 1)\,. \tag{5}$$

Hence we have an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1 (z \in \mathbb{U})$ for

$$\begin{aligned}
&\mathrm{Re}\left\{\alpha f(z) + \beta z f'(z) - \gamma \frac{zf'(z)}{f(z)} - \delta \frac{zf''(z)}{f'(z)}\right\} \\
&= \mathrm{Re}\left\{\alpha \frac{w(z)}{1-w(z)} + \beta \frac{zw'(z)}{(1-w(z))^2} - \gamma \frac{zw'(z)}{(1-w(z))w(z)} - \delta \frac{zw''(z)}{w'(z)} - 2\delta \frac{zw'(z)}{1-w(z)}\right\} \\
&> -\frac{2\alpha+\beta+2\gamma-4\delta}{4} \quad (z \in \mathbb{U})\,.
\end{aligned}$$

If there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then we have by Lemma B,

$$w(z_0) = e^{i\theta},\ \frac{z_0 w'(z_0)}{w(z_0)} = k,\ \mathrm{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq k - 1 \quad (k \geqq 1).$$

For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\mathrm{Re}\left\{\alpha f(z_0) + \beta z_0 f'(z_0) - \gamma \frac{z_0 f'(z_0)}{f(z_0)} - \delta \frac{z_0 f''(z_0)}{f'(z_0)}\right\}$$

$$
\begin{aligned}
&= \operatorname{Re}\left\{\alpha\frac{w(z_0)}{1-w(z_0)}+\beta\frac{z_0w'(z_0)}{(1-w(z_0))^2}-\gamma\frac{z_0w'(z_0)}{(1-w(z_0))w(z_0)}\right.\\
&- \left.\delta\frac{z_0w''(z_0)}{w'(z_0)}-2\delta\frac{z_0w'(z_0)}{1-w(z_0)}\right\}\\
&= \operatorname{Re}\left\{\alpha\frac{w(z_0)}{1-w(z_0)}+\beta\frac{kw(z_0)}{(1-w(z_0))^2}-\gamma\frac{kw(z_0)}{(1-w(z_0))w(z_0)}\right.\\
&- \left.\delta\frac{z_0w''(z_0)}{w'(z_0)}-2\delta\frac{kw(z_0)}{1-w(z_0)}\right\}\\
&= \operatorname{Re}\left\{(\alpha-2\delta k)\frac{w(z_0)}{1-w(z_0)}\right\}+\operatorname{Re}\left\{\frac{\beta kw(z_0)}{(1-w(z_0))^2}\right\}\\
&- \operatorname{Re}\left\{\frac{\gamma k}{1-w(z_0)}\right\}-\operatorname{Re}\left\{\frac{\delta z_0w''(z_0)}{w'(z_0)}\right\}\\
&= \frac{2\delta k-\alpha}{2}+\frac{\beta k}{2(\cos\theta-1)}-\frac{\gamma k}{2}-\operatorname{Re}\left\{\frac{\delta z_0w''(z_0)}{w'(z_0)}\right\}\\
&\leqq \frac{2\delta k-\alpha}{2}-\frac{\beta k}{4}-\frac{\gamma k}{2}+\delta(1-k)\\
&= -\frac{\alpha}{2}-\frac{(\beta+2\gamma)k}{4}+\delta\\
&\leqq -\frac{\alpha}{2}-\frac{\beta+2\gamma}{4}+\delta\\
&= -\frac{2\alpha+\beta+2\gamma-4\delta}{4}\qquad (\alpha\in\mathbb{R},\ \beta\geqq 0,\ \beta+2\gamma\geqq 0,\ \delta\geqq 0),
\end{aligned}
$$

which contradicts the hypothesis (3) of the theorem. Therefore there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus we have that

$$f(z) \prec \frac{z}{1-z},$$

which shows that

$$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\mu}d\theta \leqq \int_0^{2\pi}\left|g\left(re^{i\theta}\right)\right|^{\mu}d\theta.$$

This completes the proof.

Corollary 1. *Let the function $f(z)$ in $\mathcal{A}$ and the analytic function $g(z)$ given by (2) satisfy the conditions in Theorem 1. Then, for $\mu > 0$ and $z = r^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \frac{2\pi r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+2j)}{2^{2n}(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n}\right\}.$$

Proof.

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|\frac{re^{i\theta}}{1-re^{i\theta}}\right|^{\mu} d\theta$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \int_0^{2\pi} \left(1 - \frac{2r\cos\theta}{1+r^2}\right)^{\frac{-\mu}{2}} d\theta$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \int_0^{2\pi} \left\{\sum_{n=0}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^{n}\right\} d\theta$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \int_0^{2\pi} \left\{1 + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^{n}\right\} d\theta$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \int_0^{2\pi} \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^{n} d\theta\right\}$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(-\frac{2r}{1+r^2}\right)^{n} \int_0^{2\pi} \cos^{n}\theta \, d\theta\right\}$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{2n} \left(-\frac{2r}{1+r^2}\right)^{2n} \int_0^{2\pi} \cos^{2n}\theta \, d\theta\right\}$$

$$= \frac{r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{ 2\pi + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{2n} \left(\frac{r}{1+r^2} \right)^{2n} \cdot 2^{2n} \cdot \frac{4(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2} \right\}$$

$$= \frac{2\pi r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+2j)}{2^{2n}(2n)!} \left(\frac{r}{1+r^2} \right)^{2n} \cdot \frac{(2n)!}{(n!)^2} \right\}$$

$$= \frac{2\pi r^{\mu}}{(1+r^2)^{\frac{\mu}{2}}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+2j)}{2^{2n}(n!)^2} \left(\frac{r}{1+r^2} \right)^{2n} \right\}.$$

3 Integral means for the first derivative

The proof for the first derivative is similar.

Theorem 2. *Let $f(z)$ be in $\mathcal{A}$ and $g(z)$ be the analytic function given by (2). If the function $f(z)$ satisfies*

$$\operatorname{Re}\left\{ \alpha f'(z) + \beta \frac{zf''(z)}{f'(z)} - \gamma \frac{zf'''(z)}{f''(z)} \right\} > \frac{\alpha - 4\beta + 6\gamma}{4} \tag{6}$$

for $\alpha \geqq 0,\ 2\beta - \gamma \leqq 0$ and $\gamma \geqq 0$, then, for $\mu > 0$ and $z = re^{i\theta} (0 < r < 1)$

$$\int_0^{2\pi} \left| f'\left(re^{i\theta}\right) \right|^{\mu} d\theta \leqq \int_0^{2\pi} \left| g'\left(re^{i\theta}\right) \right|^{\mu} d\theta. \tag{7}$$

Proof. By Lemma A, it suffices to show that

$$f'(z) \prec \frac{1}{(1-z)^2}.$$

Let us define the function $w(z)$ by

$$f'(z) = \frac{1}{(1-w(z))^2} \quad (w(z) \neq 1). \tag{8}$$

Hence we have an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1 (z \in \mathbb{U})$ for

$$\begin{aligned}
&\operatorname{Re}\left\{\alpha f'(z) + \beta \frac{zf''(z)}{f'(z)} - \gamma \frac{zf'''(z)}{f''(z)}\right\} \\
&= \operatorname{Re}\left\{\alpha \frac{1}{(1-w(z))^2} + 2\beta \frac{zw'(z)}{1-w(z)} - \gamma\left(\frac{zw''(z)}{w'(z)} + \frac{3zw'(z)}{1-w(z)}\right)\right\} \\
&> \frac{\alpha - 4\beta + 6\gamma}{4} \quad (z \in \mathbb{U}).
\end{aligned}$$

If there exists $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then we have by Lemma B,

$$w(z_0) = e^{i\theta}, \; \frac{z_0 w'(z_0)}{w(z_0)} = k, \; \operatorname{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq k - 1 \quad (k \geqq 1).$$

For such a point $z_0 \in \mathbb{U}$, we obtain that

$$\begin{aligned}
&\operatorname{Re}\left\{\alpha f'(z_0) + \beta \frac{z_0 f''(z_0)}{f'(z_0)} - \gamma \frac{z_0 f'''(z_0)}{f''(z_0)}\right\} \\
&= \operatorname{Re}\left\{\alpha \frac{1}{(1-w(z_0))^2} + 2\beta \frac{z_0 w'(z_0)}{1-w(z_0)} - \gamma\left(\frac{z_0 w''(z_0)}{w'(z_0)} + \frac{3z_0 w'(z_0)}{1-w(z_0)}\right)\right\} \\
&= \operatorname{Re}\left(\frac{\alpha}{(1-w(z_0))^2}\right) + 2\operatorname{Re}\left(\frac{\beta k w(z_0)}{1-w(z_0)}\right) - \operatorname{Re}\left(\frac{\gamma z_0 w''(z_0)}{w'(z_0)}\right) - 3\operatorname{Re}\left(\frac{\gamma k w(z_0)}{1-w(z_0)}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha\cos\theta}{2(\cos\theta-1)} - \beta k + \gamma\left(-\mathrm{Re}\frac{z_0 w''(z_0)}{w'(z_0)}\right) + \frac{3}{2}\gamma k \\
&\leqq \frac{\alpha}{4} - \beta k + \gamma(1-k) + \frac{3}{2}\gamma k \\
&= \frac{\alpha}{4} + \left(\frac{\gamma}{2} - \beta\right) k + \gamma \\
&\leqq \frac{\alpha}{4} - \beta + \frac{1}{2}\gamma + \gamma \\
&= \frac{\alpha - 4\beta + 6\gamma}{4} \quad (\alpha \geqq 0,\ 2\beta - \gamma \leqq 0,\ \gamma \geqq 0),
\end{aligned}$$

which contradicts the hypothesis (6) of the Theorem 2. Therefore there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus we have that

$$f'(z) \prec \frac{1}{(1-z)^2},$$

which shows that

$$\int_0^{2\pi} \left|f'\left(re^{i\theta}\right)\right|^\mu d\theta \leqq \int_0^{2\pi} \left|g'\left(re^{i\theta}\right)\right|^\mu d\theta.$$

This completes the proof.

Corollary 2. *Let the function $f(z)$ in $\mathcal{A}$ and the analytic function g(z) given by* (2) *satisfy the conditions in Theorem 2. Then, for $\mu > 0$ and $z = r^{i\theta} (0 < r < 1)$*

$$\int_0^{2\pi} \left|f'\left(re^{i\theta}\right)\right|^\mu d\theta \leqq \frac{2\pi}{(1+r^2)^\mu}\left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+j)}{(n!)^2}\left(\frac{r}{1+r^2}\right)^{2n}\right\}.$$

Proof.

$$\int_0^{2\pi} \left|f'\left(re^{i\theta}\right)\right|^{\mu} d\theta \leqq \int_0^{2\pi} \left|\frac{1}{(1-re^{i\theta})^2}\right|^{\mu} d\theta$$

$$= \frac{1}{(1+r^2)^{\mu}} \int_0^{2\pi} \left(1 - \frac{2r\cos\theta}{1+r^2}\right)^{-\mu} d\theta$$

$$= \frac{1}{(1+r^2)^{\mu}} \int_0^{2\pi} \left\{\sum_{n=0}^{\infty} \binom{-\mu}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^n\right\} d\theta$$

$$= \frac{1}{(1+r^2)^{\mu}} \int_0^{2\pi} \left\{1 + \sum_{n=1}^{\infty} \binom{-\mu}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^n\right\} d\theta$$

$$= \frac{1}{(1+r^2)^{\mu}} \left\{2\pi + \int_0^{2\pi} \sum_{n=1}^{\infty} \binom{-\mu}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^n d\theta\right\}$$

$$= \frac{1}{(1+r^2)^{\mu}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\mu}{n} \left(-\frac{2r}{1+r^2}\right)^n \int_0^{2\pi} \cos^n\theta \, d\theta\right\}$$

$$= \frac{1}{(1+r^2)^{\mu}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\mu}{2n} \left(-\frac{2r}{1+r^2}\right)^{2n} \int_0^{2\pi} \cos^{2n}\theta \, d\theta\right\}$$

$$= \frac{1}{(1+r^2)^{\mu}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\mu}{2n} \left(\frac{r}{1+r^2}\right)^{2n} \cdot 2^{2n} \cdot \frac{4(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}\right\}$$

$$= \frac{2\pi}{(1+r^2)^{\mu}} \left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+j)}{(2n)!} \left(\frac{r}{1+r^2}\right)^{2n} \cdot \frac{(2n)!}{(n!)^2}\right\}$$

$$= \frac{2\pi}{(1+r^2)^{\mu}} \left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+j)}{(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n}\right\}.$$

Putting $\mu = 1$ in Corollary 2, we have the following.

Corollary 3. *Let the function* $f(z)$ *in* $\mathcal{A}$ *and the analytic function* $g(z)$ *given by* (2) *satisfy the conditions in Theorem 2. Then, for* $0 < r < 1$

$$|f(z)| \leqq \frac{2\pi}{1-r^2}.$$

Proof.

$$\begin{aligned}
|f(z)| &\leqq \frac{2\pi}{1+r^2}\left\{1+\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{2n-1}(1+j)}{(n!)^2}\left(\frac{r}{1+r^2}\right)^{2n}\right\} \\
&= \frac{2\pi}{1+r^2}\left\{1+\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}\left(\left(\frac{r}{1+r^2}\right)^{2}\right)^{n}\right\} \\
&= \frac{2\pi}{1+r^2}\left\{1+\sum_{n=1}^{\infty}\frac{2^{2n}\left(\frac{1}{2}\right)_n(1)_n}{((1)_n)^2}\left(\left(\frac{r}{1+r^2}\right)^{2}\right)^{n}\right\} \\
&= \frac{2\pi}{1+r^2}\left\{1+\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n(1)_n}{((1)_n)^2}\left(\left(\frac{2r}{1+r^2}\right)^{2}\right)^{n}\right\} \\
&= \frac{2\pi}{1+r^2}F\left(\frac{1}{2},1;1;\left(\frac{2r}{1+r^2}\right)^{2}\right) \\
&= \frac{2\pi}{1+r^2}\left\{1-\left(\frac{2r}{1+r^2}\right)^{2}\right\}^{-\frac{1}{2}} \\
&= \frac{2\pi}{1+r^2}\left(\frac{1-r^2}{1+r^2}\right)^{-1} \\
&= \frac{2\pi}{1-r^2}.
\end{aligned}$$

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