Bounds for the Čebyšev Functional of a Convex and a Bounded Function

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Dedicated to Professor Ph.D. Alexandru Lupaş on the occasion of his 65th birthday

Abstract

Upper and lower bounds for the Čebyšev functional of a convex and a bounded function are given. Some applications for quadrature rules and probability density functions are also provided.

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1 Introduction

For two Lebesgue functions \( f, g : [a, b] \rightarrow \mathbb{R} \), consider the Čebyšev functional

\[
C (f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt.
\]

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In 1971, F.V. Atkinson [1] showed that if \( f, g \) are twice differentiable and convex on \([a, b]\) and
\[
\int_a^b \left( t - \frac{a + b}{2} \right) g(t) \, dt = 0,
\]
then \( C(f, g) \) is nonnegative.

This result is, in fact, implied by that of A. Lupas [3] who proved that for any two convex functions \( f, g : [a, b] \to \mathbb{R} \) the lower bound for the Čebyšev functional is:
\[
C(f, g) \geq \frac{12}{(b-a)^3} \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt \cdot \int_a^b \left( t - \frac{a + b}{2} \right) g(t) \, dt,
\]
with true equality holding when at least one of \( f \) or \( g \) is a linear function on \([a, b]\).

As pointed out in [4, p. 262], if the functions \( f, g \) are convex and one is symmetric, then \( C(f, g) \geq 0 \).

For other results for convex integrands, see [4, p. 256] and [4, p. 262] where further references are given.

In this note we provide some bounds for the Čebyšev functional in the case of a convex function \( g \) and a bounded function \( f \). Some applications are given as well.

2 The Results

For an integrable function \( f : [a, b] \to \mathbb{R} \), define the \((\gamma - 2)\)–moment by
\[
M_{2,\gamma} (f) := \int_a^b (t - \gamma)^2 f(t) \, dt.
\]

For a convex function \( g : [a, b] \to \mathbb{R} \) for which the derivatives \( g_-'(b) \) and \( g_+'(a) \) are finite, define
\[
\Gamma(f, g) := \frac{g'_-(b) M_{2,b} (f) - g'_+(a) M_{2,a} (f)}{2 (b-a)^2},
\]
where \( f \) is integrable on \([a, b]\).

The following result holds:

**Theorem 2.1.** If \( f : [a, b] \to \mathbb{R} \) is a Lebesgue measurable function such that there exist constants \( m, M \in \mathbb{R} \) with

\[
m \leq f(t) \leq M \quad \text{for a.e. } t \in [a, b],
\]

and \( g : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\) with the lateral derivatives \( g'_+ (a) \) and \( g'_- (b) \) finite, then,

\[
\frac{1}{6} m (b - a) \left[ g'_- (b) - g'_+ (a) \right] - \Gamma (f, g) \leq C (f, g) \leq \frac{1}{6} M (b - a) \left[ g'_- (b) - g'_+ (a) \right] - \Gamma (f, g).
\]

**Proof.** We use Sonin’s identity [4, p. 246]:

\[
C (f, g) = \frac{1}{b - a} \int_a^b (f(t) - \gamma) \left( g(t) - \frac{1}{b - a} \int_a^b g(s) \, ds \right) \, dt,
\]

for any \( \gamma \in \mathbb{R} \), and the following inequality for convex functions obtained by S.S. Dragomir in [2]:

\[
\frac{1}{b - a} \int_a^b g(s) \, ds - g(t) \leq \frac{1}{2 (b - a)} \left[ (b - t)^2 g'_- (b) - (t - a)^2 g'_+ (a) \right]
\]

for any \( t \in [a, b] \). The constant \( \frac{1}{2} \) is sharp.

By Sonin’s identity for \( \gamma = M \), we have,

\[
C (f, g) = \frac{1}{b - a} \int_a^b (M - f(t)) \left( \frac{1}{b - a} \int_a^b g(s) \, ds - g(t) \right) \, dt.
\]

From (7) we get,

\[
\left( \frac{1}{b - a} \int_a^b g(s) \, ds - g(t) \right) (M - f(t)) \leq
\]
\[
\frac{1}{2(b-a)} \left[ g_-'(b) (b-t)^2 (M-f(t)) - g_+'(a) (t-a)^2 (M-f(t)) \right]
\]
for a.e. \( t \in [a,b] \).

Integrating (9) over \( t \) on \([a,b]\) and using the representation (8), we get

\[
C(f, g) \leq \frac{1}{2(b-a)^2} \left[ M \int_a^b \left[ g_-'(b) (b-t)^2 - g_+'(a) (t-a)^2 \right] dt - g_-'(b) \int_a^b (b-t)^2 f(t) dt + g_+'(a) \int_a^b (t-a)^2 f(t) dt \right].
\]

Since
\[
\int_a^b \left[ g_-'(b) (b-t)^2 - g_+'(a) (t-a)^2 \right] dt = \frac{(b-a)^3}{3} \left[ g_-'(b) - g_+'(a) \right]
\]
then (10) provides the second part of (5).

Again, by Sonin’s identity,
\[
C(f, g) = \frac{1}{b-a} \int_a^b (m-f(t)) \left( \frac{1}{b-a} \int_a^b g(s) ds - g(t) \right) dt.
\]

Utilising (7) and the fact that \( m-f(t) \leq 0 \) for a.e. \( t \in [a,b] \), we obtain,

\[
C(f, g) \geq \frac{1}{2(b-a)^2} \int_a^b \left[ (b-t)^2 g_-'(b) (m-f(t)) - (t-a)^2 g_+'(a) (m-f(t)) \right] dt = \frac{1}{2(b-a)^2} \left[ m \int_a^b [(b-t)^2 g_-'(b) - (t-a)^2 g_+'(a)] dt - 2(b-a) \Gamma(f,g) \right],
\]
giving the first part of (5).

The following particular result holds.

**Corollary 2.1.** Let \( f : [a,b] \to \mathbb{R} \) be an essentially bounded function on \([a,b]\), i.e., \( f \in L_\infty[a,b] \) and \( \|f\|_\infty := \text{ess} \sup_{t \in [a,b]} |f(t)| \), its norm. If \( g : [a,b] \to \mathbb{R} \) is a convex function on \([a,b]\) with the lateral derivatives \( g_+'(a) \) and \( g_-'(b) \) finite, then we have the inequality:

\[
|C(f, g) + \Gamma(f, g)| \leq \frac{1}{6} \|f\|_\infty (b-a) \left[ g_-'(b) - g_+'(a) \right].
\]
3  Applications for the Trapezoid Rule

The following result is a perturbed version of the trapezoid rule.

**Proposition 3.1.** Let \( h : [a, b] \to \mathbb{R} \) be a differentiable function with the property that the derivative \( h' : (a, b) \to \mathbb{R} \) is convex on \((a, b)\). If \( h''_+(a)\), \( h''_-(b)\) are finite, then,

\[
\left| \frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(t) \, dt - \frac{(b-a)^2}{12} \cdot \frac{h''_+(a) + h''_-(b)}{2} \right| \leq \frac{1}{12} (b-a)^2 \cdot \left[ h''_-(b) - h''_+(a) \right].
\]

**Proof.** Consider the functions \( f, g : [a, b] \to \mathbb{R} \) defined by \( f(t) = t - \frac{a+b}{2} \), \( g(t) = h'(t) \). For these functions, a simple calculation shows that, \( \Gamma(f, g) = -\frac{(b-a)^2}{12} \cdot \frac{h''_+(a) + h''_-(b)}{2} \), since,

\[
\int_a^b (t-b)^2 \left( t - \frac{a+b}{2} \right) dt = -\frac{(b-a)^4}{12}
\]

and \( \int_a^b (t-a)^2 \left( t - \frac{a+b}{2} \right) dt = \frac{(b-a)^4}{12} \).

Clearly, also, \( \|f\|_\infty = \frac{1}{2} (b-a) \). Utilising the elementary identity,

\[
\frac{1}{b-a} \int_a^b \left( t - \frac{a+b}{2} \right) h'(t) \, dt = \frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(t) \, dt
\]

and the fact that, for \( f, g \) as defined previously,

\[
C(f, g) = \frac{1}{b-a} \int_a^b \left( t - \frac{a+b}{2} \right) h'(t) \, dt,
\]

that a direct application of Corollary 2.1 reveals the desired inequality (12).
Remark 3.1. Similar results may be stated if one considers quadrature rules for which the remainder \( R(f) \) can be expressed in Peano kernel form, i.e., \( R(f) = \int_a^b K(t) f^{(n)}(t) \, dt \), where \( K(t) \) is a kernel for which the supremum norm can be easily computed and the \( n \)–th derivative of the function, \( f \), is assumed to be convex on \( (a, b) \). The exploration of these bounds is left to the interested reader.

4 Applications for Probability Density Functions

Let \( f : [a, b] \to [0, \infty) \) be a density function, this means that \( f \) is integrable on \( [a, b] \) and \( \int_a^b f(t) \, dt = 1 \) and let \( F(x) := \int_a^x f(t) \, dt, \quad x \in [a, b] \) be its distribution function. We also denote the expectation of \( f \) by \( E(f) \), where \( E(f) := \int_a^b tf(t) \, dt \), provided the integral exists and is finite, and the mean deviation \( M_D(f) \), by, \( M_D(f) := \int_a^b |t - E(f)|f(t) \, dt \).

Theorem 4.1. Let \( f : [a, b] \to [0, \infty) \) be a density function with the property that there exists \( m, M \geq 0 \) such that \( m \leq f(t) \leq M \) for a.e. \( t \in [a, b] \) then,

\[
\frac{1}{3} m (b-a)^2 \leq M_D(f) + \frac{1}{b-a} M_2 \frac{a+b}{2} (f) - \frac{(E(f) - \frac{a+b}{2})^2}{b-a} \leq \frac{1}{3} M (b-a)^2.
\]

Proof. We apply Theorem 2.1 for \( g : [a, b] \to \mathbb{R}, g(t) = |t - E(f)| \). Since \( g^- (b) = 1, \quad g^+ (a) = -1 \), then,

\[
\Gamma(f, g) = \frac{1}{(b-a)^2} \int_a^b \left[ \frac{(t-a)^2 + (t-b)^2}{2} \right] f(t) \, dt =
\]
On the other hand,

\[
C(f, g) = \frac{1}{b-a} \int_a^b |t - E(f)| f(t) \, dt - \frac{1}{b-a} \int_a^b |t - E(f)| \, dt \cdot \frac{1}{b-a} \int_a^b f(t) \, dt =
\]

\[
= \frac{1}{b-a} M_D(f) - \frac{1}{(b-a)^2} \left[ \frac{(b - E(f))^2 + (E(f) - a)^2}{2} \right] =
\]

\[
= \frac{1}{b-a} M_D(f) - \frac{1}{(b-a)^2} \left[ \left( E(f) - \frac{a+b}{2} \right)^2 + \frac{1}{4} (b-a)^2 \right] =
\]

\[
= \frac{1}{b-a} M_D(f) - \frac{(E(f) - \frac{a+b}{2})^2}{(b-a)^2} - \frac{1}{4}.
\]

Making use of the inequality (5) we deduce the desired result (13).

If one is interested in providing bounds for the absolute moment around the midpoint \( \frac{a+b}{2} \), \( M_{2, \frac{a+b}{2}}(f) := \int_a^b \left| t - \frac{a+b}{2} \right| f(t) \, dt \), then on applying Theorem 2.1 for \( g(t) = \left| t - \frac{a+b}{2} \right| \), we have the following.

**Theorem 4.2.** Let \( f : [a, b] \to [0, \infty) \) be as in Theorem 4.1, then

\[
\frac{1}{3} m(b-a)^2 \leq M_{2, \frac{a+b}{2}}(f) + \frac{1}{b-a} M_{2, \frac{a+b}{2}}(f) \leq \frac{1}{3} M(b-a)^2.
\]

**Remark 4.1.** Similar results may be stated if one considers higher moments

\[
M_{p, \gamma}(f) := \int_a^b |t - \gamma|^p f(t) \, dt, \quad p \geq 1,
\]

for which \( g(t) = |t - \gamma|^p \) in Theorem 2.1 will procure the corresponding bounds in terms of \( m \) and \( M \) with the property that \( 0 < m \leq f(t) \leq M \) for a.e. \( t \in [a, b] \). The details are omitted.
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