Rate of convergence on the mixed summation integral type operators

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

Gupta and Erkus [3] introduced the mixed sequence of summation-integral type operators $S_n(f, x)$ and estimated some direct results in simultaneous approximation. We extend the study on these operators $S_n(f, x)$ and here we study the rate of convergence for functions having derivatives of bounded variation.

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1 Introduction

Very recently Gupta and Erkus [3] defined a mixed summation-integral type operators to approximate integrable functions on the interval $[0, \infty)$. The operators introduced in [3] are defined as

$$S_n(f, x) = \int_0^\infty W_n(x, t)f(t)dt$$

$$= (n - 1) \sum_{0}^{\infty} s_{n,v}(x) \int_0^{\infty} b_{n,v-1}(t)f(t)dt + \exp (-nx)f(0), \quad x \in [0, \infty)$$

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29
where
\[ W_n(x, t) = (n - 1) \sum_{v=1}^{\infty} s_{n,v}(x) b_{n,v-1}(t) + \exp^{-nx} \delta(t) \]
\( \delta(t) \) being Dirac delta function, \( s_{n,v}(x) = \frac{\exp(-nxv)}{v!} \) and \( b_{n,v}(t) = \left( \frac{n+v-1}{v} \right) t^v(1+t)^{-n-v} \) are respectively Szasz and Baskakov basis functions. Although these operators are similar to the generalized summation integral type operators recently introduced by Srivastava and Gupta [4], but the approximation properties of the operators (1) are different from those introduced in [4]. Here in summation and integration we are taking different basis functions.

We define
\[ \beta_n(x, t) = \int_{0}^{t} W_n(x, s) ds \]
then in particular,
\[ \beta_n(x, \infty) = \int_{0}^{\infty} W_n(x, s) ds = 1. \]

Let \( DB_\gamma(0, \infty), \gamma \geq 0 \) be the class of absolutely continuous functions \( f \) defined on \((0, \infty)\) satisfying the growth condition \( f(t) = O(t^\gamma), t \to \infty \) and having a derivative \( f' \) on the interval \((0, \infty)\) coinciding a.e. with a function which is of bounded variation on every finite subinterval of \((0, \infty)\). It can be observed that all functions \( f \in BD_\gamma(0, \infty) \) posses for each \( c > 0 \) a representation
\[ f(x) = f(c) + \int_{c}^{x} \psi(t) dt, \quad x \geq c. \]
We denote the auxiliary function \( f_x \) by
\[ f_x(t) = \begin{cases} 
  f(t) - f(x^-), & 0 \leq t < x; \\
  0, & t = x; \\
  f(t) - f(x^+), & x < t < \infty.
\end{cases} \]
In [3] the authors studied some direct results in simultaneous approximation for the operators (1). The rates of convergence for functions having derivatives of bounded variation on Bernstein polynomials were studied in [1] and
This motivated us to study further on summation integral type operators and here we estimate the rate of convergence for the operator (1) with functions having derivatives of bounded variation.

## 2 Auxiliary Results

We shall use the following Lemmas to prove our main theorem.

**Lemma 2.1.** Let the function \( \mu_{n,m}(x) \), \( m \in \mathbb{N}^0 \), be defined as

\[
\mu_{n,m}(x) = (n - 1) \sum_{v=1}^{\infty} s_{n,v}(x) \int_{0}^{\infty} b_{n,v-1}(t)(t - x)^m dt + (-x)^m \exp(-nx).
\]

Then \( \mu_{n,0}(x) = 1 \), \( \mu_{n,1}(x) = \frac{2x}{n-2} \), \( \mu_{n,2}(x) = \frac{nx(x+2)+6x}{(n-2)(n-3)} \), also we have the recurrence relation:

\[
(n - m - 2)\mu_{n,m+1}(x) = x[\mu_{n,m}^{(1)}(x) + m(x + 2)\mu_{n,m-1}(x)] + [m + 2x(m + 1)]\mu_{n,m}(x); \quad n > m + 2.
\]

Consequently for each \( x \in [0, \infty) \) we from this recurrence relation that

\[
\mu_{n,m}(x) = O(n^{-(m+1)/2}).
\]

**Remark 2.2.** In particular given any number \( \lambda > 1 \) and \( x \in (0, \infty) \), by lemma 2.1, we have for \( n \) sufficiently large

\[
S_n((t-x)^2, x) \equiv \mu_{n,2} \leq \frac{\lambda x(x+2)}{n}
\]

**Remark 2.3.** From equation (2) it follows that

\[
S_n(|t-x|, x) \leq [S_n((t-x)^2, x)]^{1/2} \leq \sqrt{\lambda x(x+2)/n}
\]

**Lemma 2.4.** Let \( x \in (0, \infty) \) and \( W_n(x,t) \) are as in (1). Then for \( \lambda > 1 \) and for \( n \) sufficiently large, we have

\[
(i) \quad \beta_n(x,y) = \int_{0}^{y} W_n(x,t) dt \leq \frac{\lambda x(x+2)}{n(x-y)^2}, \quad 0 \leq y < x
\]

\[
(ii) \quad 1 - \beta_n(x,z) = \int_{z}^{\infty} W_n(x,t) dt \leq \frac{\lambda x(x+2)}{n(z-x)^2}, \quad x < z < \infty
\]
Proof. First we prove (i), by (2), we have
\[
\int_0^y W_n(x, t) dt \leq \int_0^y \frac{(x-t)^2}{(x-y)^2} W_n(x, t) dt \\
\leq (x,y)^{-2} \mu_n(x) \leq \frac{\lambda(x+2)}{x(x-y)^2}.
\]
The proof of (ii) is similar, we omit the details.

3 Main Result

In this section, we prove the following main theorem.

**Theorem 3.1** Let \( f \in DB_\gamma(0, \infty), \gamma > 0 \), and \( x \in (0, \infty) \). Then for \( \lambda > 2 \) and for \( n \) sufficiently large, we have
\[
| S_n(f, x) - f(x) | \leq \frac{\lambda(x+2)}{n} \left( \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \left( \int_{x-x/k}^{x+x/k} \right) + \frac{x}{\sqrt{n}} \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} ((f')_x) \right)
\]
\[
+ \frac{\lambda(x+2)}{n} (| f(2x) - f(x) - x f'(x^+) | + | f(x) |)
\]
\[
+ \sqrt{\lambda x(x+2)/n} (M2^\gamma O(n^{-2}) + | f(x^+) |)
\]
\[
+ 1/2 \sqrt{\lambda x(x+2)/n} | f'(x^+) | - | f'(x^-) |
\]
\[
+ \frac{x}{n-2} | f'(x^+) | + | f'(x^-) |,
\]
where \( \int_{a}^{b} f(x) \) denotes the total variation of \( f_x \) on \( [a,b] \).

**Proof.** We have
\[
S_n(f, x) - f(x) = \int_0^\infty W_n(x, t)(f(t) - f(x)) dt
\]
\[
= \int_0^\infty \left( \int_x^t W_n(x, t)(f'(u) du) dt \right)
\]
Using the identity
\[
f'(u) = 1/2[f'(x^+) + f'(x^-)] + (f')_x(u) + 1/2[f'(x^+) - f'(x^-)] sgn(u-x)
\]
Rate of convergence on the mixed summation integral type operators

\[ f'(x) - 1/2[f'(x^+) + f'(x^-)] \chi(x(u)), \]

it is easily verified that

\[ \int_0^\infty \left( \int_x^t f'(x) - 1/2[f'(x^+) + f'(x^-)] \chi(x(u)) du \right) W_n(x, t) dt = 0, \]

Also

\[ \int_0^\infty \left( \int_x^t 1/2[f'(x^+) - f'(x^-)] sgn(u - x) du \right) W_n(x, t) dt \]

\[ = 1/2[f'(x^+) - f'(x^-)] S_n(|t - x|, x) \]

and

\[ \int_0^\infty \left( \int_x^t 1/2[f'(x^+) + f'(x^-)] du \right) W_n(x, t) dt \]

\[ = 1/2[f'(x^+) + f'(x^-)] S_n((t - x), x). \]

Thus we have

\begin{align*}
|S_n(f, x) - f(x)| &\leq |\int_0^\infty \left( \int_x^t (f')_x(u) du \right) W_n(x, t) dt - \int_x^x \left( \int_x^t (f')_x(u) du \right) W_n(x, t) dt| \\
&\quad + \frac{1}{2} |f'(x^+) - f'(x^-)| S_n(|t - x|, x) \\
&\quad + \frac{1}{2} |f'(x^+) + f'(x^-)| S_n((t - x), x) \\
&= |A_n(f, x) + B_n(f, x)| + \frac{1}{2} |f'(x^+) - f'(x^-)| S_n(|t - x|, x) \\
&\quad + \frac{1}{2} |f'(x^+) + f'(x^-)| S_n((t - x), x). \end{align*}

To complete the proof of the theorem it is sufficient to estimate the terms \(A_n(f, x)\) and \(B_n(f, x)\). Applying integration by parts, using Lemma 2.4 and taking \(y = x - x/\sqrt{n}\), we have

\[ |B_n(f, x)| = |\int_0^x (\int_x^t (f')_x(u) du) dt | \beta_n(x, t) dt | \]
\[ \int_0^x \beta_n(x, t)(f')_x(t)dt \leq \left( \int_0^y + \int_y^x \right) |(f')_x(t)| \beta_n(x, t) | dt \]
\[ \leq \frac{\lambda x(x+2)}{n} \int_0^y \sqrt{\frac{x}{(x-t)^2}} dt + \int_y^x \sqrt{\frac{x}{(x-t)^2}} dt \]
\[ \leq \frac{\lambda x(x+2)}{n} \int_0^y \sqrt{\frac{x}{(x-t)^2}} dt + \frac{x}{\sqrt{n}} \sqrt{\frac{1}{x-\frac{x}{n}}} \sqrt{(f')_x}. \]

Let \( u = \frac{x}{x-t} \). Then we have
\[ \frac{\lambda x(x+2)}{n} \int_0^y \sqrt{\frac{x}{(x-t)^2}} dt = \frac{\lambda x(x+2)}{n} \int_{\frac{x}{x-u}}^1 \sqrt{\frac{x}{x}} du \]
\[ \leq \frac{\lambda x(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{\frac{x}{x-\frac{x}{n}}} \sqrt{(f')_x}. \]

Thus
\[ | \beta_n(f, x) | \leq \frac{\lambda x(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \sqrt{\frac{x}{x-\frac{x}{n}}} \sqrt{(f')_x} + \frac{x}{\sqrt{n}} \sqrt{\frac{1}{x-\frac{x}{n}}} \sqrt{(f')_x}. \]

On the other hand, we have
\[ | A_n(f, x) | = | \int_x^{\infty} (\int_x^t (f')_x(u)du)W_n(x, t)dt | \]
\[ = | \int_{2x}^{\infty} (\int_x^t (f')_x(u)du)W_n(x, t)dt + \int_x^{2x} (\int_x^t (f')_x(u)du)dt(1 - \beta_n(x, t)) | dt \]
\[ \leq | \int_{2x}^{\infty} (f(t) - f(x))W_n(x, t)dt | + | f'(x^+) | \int_{2x}^{\infty} (t-x)W_n(x, t)dt | \]
\[ + | \int_x^{2x} (f')_x(u)du | | (1 - \beta_n(x, 2x)) | + \int_x^{2x} \sqrt{(f')_x(t)} | | (1 - \beta_n(x, t)) | dt \]
\[ \leq \frac{M}{x} \int_{2x}^{\infty} W_n(x, t) t^\gamma | t - x | dt + \frac{| f(x) |}{x^2} \int_{2x}^{\infty} W_n(x, t)(t-x)^2 dt \]
Rate of convergence on the mixed summation integral type operators

\[ + |f'(x^+)| \int_{2x}^{\infty} W_n(x, t) |(t - x)| dt + \frac{\lambda(x + 2)}{nx} (|f(2x) - f(x) - xf'(x^+)|

\[ + \frac{\lambda(x + 2)}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor \frac{x + \frac{x}{\sqrt{n}}} {x} \sqrt{((f')_x)} + \frac{x + \frac{xf'(x^+)}{\sqrt{n}}} {x} \sqrt{((f')_x)}.\]

Next applying H"{o}lder’s inequality, and Lemma 2.1, we proceed as follows for the estimation of the first two terms in the right hand side of (6):

\[ (7) \quad \frac{M}{x} \int_{2x}^{\infty} W_n(x, t) t^\gamma |t - x| dt + \frac{|f(x)|}{x^2} \int_{2x}^{\infty} W_n(x, t) (t - x)^2 dt \]

\[ \leq \frac{M}{x} \left( \int_{2x}^{\infty} W_n(x, t) t^{2\gamma} dt \right)^{\frac{1}{2}} + \left( \int_{2x}^{\infty} W_n(x, t) (t - x)^2 dt \right)^{\frac{1}{2}} \]

\[ + \frac{|f(x)|}{x^2} \left( \int_{2x}^{\infty} W_n(x, t) (t - x)^2 dt \right)^{\frac{1}{2}} \]

\[ \leq M2^\gamma O(n^{-\gamma/2}) \frac{\sqrt{\lambda x(x + 2)}}{\sqrt{n}} + |f(x)| \frac{\lambda(x + 2)}{nx} \]

Also the third term of the right side of (6) is estimated as

\[ |f'(x^+)| \int_{2x}^{\infty} W_n(x, t) |t - x| dt \]

\[ \leq |f'(x^+)| \int_{2x}^{\infty} W_n(x, t) |t - x| dt \]

\[ \leq |f'(x^+)| \left( \int_{0}^{\infty} W_n(x, t) (t - x)^2 dt \right)^{\frac{1}{2}} \left( \int_{0}^{\infty} W_n(x, t) dt \right)^{\frac{1}{2}} \]

\[ = |f'(x^+)| \frac{\sqrt{\lambda x(x + 2)}}{\sqrt{n}} \]

Combining the estimates (4)-(7), we get the desired result.

This completes the proof of Theorem 3.1.
References


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