A New Class of Multivalent Harmonic Functions

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In memoriam of Associate Professor Ph. D. Luciana Lupaș

Abstract

In this paper, we introduce a new class of multivalent harmonic functions. We investigate various properties of functions belonging to this class. Coefficients bounds, distortion bounds and extreme points are given.

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1 Introduction

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both $u$ and $v$ are real harmonic in $\mathbb{C}$. In any

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simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$. See Clunie and Sheil-Small (see [2]).

Denote by $H(p)$ the class of functions $f = h + \overline{g}$ that are harmonic multivalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. For $f = h + \overline{g} \in H(p)$ we may express the analytic functions $h$ and $g$ as

\begin{equation}
(1) \quad h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p}^{\infty} b_k z^k, \quad |b_p| < 1.
\end{equation}

Also denote by $T(p)$, the subclass of $H(p)$ consisting of all functions $f = h + \overline{g}$ where $h$ and $g$ are given by

\begin{equation}
(2) \quad h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k, \quad g(z) = -\sum_{k=p}^{\infty} |b_k| z^k, \quad |b_p| < 1.
\end{equation}

We denote by $H^p_{\lambda}(p, \alpha)$ the class of all functions of the form (1.1) that satisfy the condition

\begin{equation}
(3) \quad \Re \left\{ \frac{(D_{\lambda}^{n+p-1}f(z))'}{pz_{p-1}} \right\} > \alpha,
\end{equation}

where $0 \leq \alpha < p$, $p \in \mathbb{N}$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $D_{\lambda}^{n+p-1}f(z) = D_{\lambda}^{n+p-1}h(z) + D_{\lambda}^{n+p-1}g(z)$.

When $p = 1$, $D_{\lambda}^{n}$ denotes the operator introduced by [3]. For $h$ and $g$ given by (1.1) we have

\[ D_{\lambda}^{n+p-1}h(z) = z^p + \sum_{k=p+1}^{\infty} \left[ 1 + \lambda(k - p) \right] C(n, k, p) a_k z^k, \]
\[ D^{n+p-1}_\lambda g(z) = \sum_{k=p}^{\infty} \left[ 1 + \lambda (k - p) \right] C(n, k, p) b_k z^k \]

where \( \lambda \geq 0, p \in \mathbb{N}, n > -p \) and \( C(n, k, p) = \binom{k + n - 1}{n + p - 1} \).

Note that :

- \( \mathcal{H}_0^0(1,0) \equiv S^*_H \) studied by Silverman [1],
- \( \mathcal{H}_0^0(1,0) \equiv H(\lambda) \) studied by Yalçın and Öztürk [7],
- \( \mathcal{H}_0^0(1,\alpha) \equiv N_H(\alpha) \) studied by Ahuja and Jahangiri [5],
- \( \mathcal{H}_0^0(1,0) \equiv \mathcal{H}_0^\lambda(p,\alpha) \) studied by the authors in [4].

Also we note that for the analytic part the class \( \mathcal{H}_0^n(p,\alpha) \) was introduced and studied by Goel and Sohi [6].

We further denote by \( T_n^\alpha(p,\alpha) \) the subclass of \( \mathcal{H}_0^n(p,\alpha) \), where

\[ T_n^\alpha(p,\alpha) = T(p) \cap \mathcal{H}_0^n(p,\alpha). \]

### 2 Coefficients Bounds

**Theorem 2.1.** Let \( f = h + \bar{g} \) with \( h \) and \( g \) are given by (1.1). Let

\[ \sum_{k=p}^{\infty} k \left[ 1 + \lambda (k - p) \right] C(n, k, p) |a_k| + |b_k| \leq p(2 - \alpha) \]

where \( a_p = p, \lambda \geq 0 \) and \( 0 \leq \alpha < p \). Then \( f \) is harmonic multivalent sense preserving in \( \mathbb{U} \) and \( f \in \mathcal{H}_0^n(p,\alpha) \).

**Proof.** Letting \( w(z) = \frac{(D^{n+p-1}_\lambda f(z))'}{p^{p-1}} \). Using the fact \( \Re\{w\} \geq \alpha \) if and only
if $|p - \alpha + w(z)| \geq |p + \alpha - w(z)|$, it suffices to show that

$$|p - \alpha + \frac{(D^{n+p-1}_\lambda f(z))'}{p^{2p-1}} - p + \alpha - \frac{(D^{n+p-1}_\lambda f(z))'}{p^{2p-1}}| \geq 0. \tag{5}$$

Substituting for $h$ and $g$ in (2.2) yields

$$|p - \alpha + \frac{(D^{n+p-1}_\lambda h(z))'}{p^{2p-1}} + \frac{(D^{n+p-1}_\lambda g(z))'}{p^{2p-1}} - p + \alpha - \frac{(D^{n+p-1}_\lambda h(z))'}{p^{2p-1}} - \frac{(D^{n+p-1}_\lambda g(z))'}{p^{2p-1}}| =$$

$$= \left| p + 1 - \alpha + \sum_{k=p+1}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k - p) \right] C(n, k, p) a_k z^{k-p} + \right.$$ 

$$+ \sum_{k=p}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k - p) \right] C(n, k, p) b_k z^{k-p} \right| -$$ 

$$- \left| p - 1 + \alpha - \sum_{k=p+1}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k - p) \right] C(n, k, p) a_k z^{k-p} - \right.$$ 

$$- \sum_{k=p}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k - p) \right] C(n, k, p) b_k z^{k-p} \right| \geq$$

$$\geq 2 \left\{ (1 - \alpha) - \left[ \sum_{k=p+1}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k - p) \right] C(n, k, p) |a_k|| z^{k-p}| + \right.$$ 

$$+ \sum_{k=p}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k - p) \right] C(n, k, p) |b_k|| z^{k-p}| \right] \right\} >$$

$$> 2 \left\{ p(1 - \alpha) - \left[ \sum_{k=p+1}^{\infty} k \left[ 1 + \lambda(k - p) \right] C(n, k, p) |a_k| + \right.$$ 

$$+ \sum_{k=p}^{\infty} k \left[ 1 + \lambda(k - p) \right] C(n, k, p) |b_k| \right] \right\} > 0.
The Harmonic mappings

\[ f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{x_k}{k[1 + \lambda(k - p)]C(n, k, p)}z^k + \]
\[ + \sum_{k=p}^{\infty} \frac{\overline{y}_k}{k[1 + \lambda(k - p)]C(n, k, p)}z^k \]

where \( \sum_{k=p+1}^{\infty} |x_k| + \sum_{k=p}^{\infty} |y_k| = p(1 - \alpha) \), show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in \( H_\lambda^n(p, \alpha) \) because

\[ \sum_{k=p+1}^{\infty} k[1 + \lambda(k - p)]C(n, k, p)(|a_k| + |b_k|) = \]
\[ = p + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = p(2 - \alpha). \]

The restriction placed in Theorem 2.1 on the moduli of the coefficients of \( f = h + \overline{g} \) enables us to conclude for arbitrary rotation of the coefficients of \( f \) that the resulting functions would still be harmonic multivalent and \( f \in H_\lambda^n(p, \alpha) \).

We next show that the condition (2.1) is also necessary for functions in \( T_\lambda^n(p, \alpha) \).

**Theorem 2.2.** Let \( f = h + \overline{g} \) with \( h \) and \( g \) are given by (1.2). Then \( f \in T_\lambda^n(p, \alpha) \) if and only if

\[ \sum_{k=p}^{\infty} k[1 + \lambda(k - p)]C(n, k, p)[|a_k| + |b_k|] \leq p(2 - \alpha) \]  

where \( a_p = p, \lambda \geq 0 \) and \( 0 \leq \alpha < p \).
Proof. The "if" part follows from Theorem 2.1 upon noting $T_{\lambda}^n(p, \alpha) \subset \mathcal{H}_{\lambda}^n(p, \alpha)$. For the "only if" part, assume that $f \in T_{\lambda}^n(p, \alpha)$. Then by (1.3) we have
\[
\Re \left\{ \frac{(D_{\lambda}^n h(z))' + (D_{\lambda}^n g(z))'}{pz^{p-1}} \right\} =
\Re \left\{ 1 - \sum_{k=p+1}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k-1) \right] C(n, k) |a_k| z^{k-p} - \sum_{k=p}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k-1) \right] C(n, k) |b_k| z^{k-p} \right\} > \alpha.
\]
If we choose $z$ to be real and let $z \to 1^-$, we get
\[
1 - \sum_{k=p+1}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k-1) \right] C(n, k) |a_k| z^{k-p} - \sum_{k=p}^{\infty} \frac{k}{p} \left[ 1 + \lambda(k-1) \right] C(n, k) |b_k| z^{k-p} \geq \alpha,
\]
which is precisely the assertion (2.4) of Theorem 2.2.

3 Distortion Bounds and Extreme Points.

In this section, we shall obtain distortion bounds for functions in $T_{\lambda}^n(p, \alpha)$ and also provide extreme points for the class $T_{\lambda}^n(p, \alpha)$.

Theorem 3.1. If $f \in T_{\lambda}^n(p, \alpha)$, for $\lambda \geq 0$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $|z| = r > 1$, then
\[
|f(z)| \leq (1 + b_p)r^p + \frac{p(1 - \alpha) - |b_p|}{(p + 1)(1 + \lambda)(n + p)}r^{p+1},
\]
and
\[
|f(z)| \geq (1 - b_p)r^p - \frac{p(1 - \alpha) - |b_p|}{(p + 1)(1 + \lambda)(n + p)}r^{p+1}.
\]
Proof. We only prove the second inequality. The argument for first inequality is similar and will be omitted. Let \( f \in T^n_\lambda(p, \alpha) \). Taking the absolute value of \( f \), we obtain

\[
|f(z)| \geq (1 - b_p)r^p - \sum_{k=p+1}^{\infty} (|a_k| + |b_k|)r^k \geq (1 - b_p)r^p - \sum_{k=p+1}^{\infty} (|a_k| + |b_k|)r^{p+1} = (1 - b_p)r^p - \frac{1}{(p + 1)(1 + \lambda)(n + p)}. \]

\[
\cdot \sum_{k=p+1}^{\infty} (p + 1)(1 + \lambda)(n + p)(|a_k| + |b_k|)r^{p+1} \geq \geq (1 - b_p)r^p - \frac{1}{(p + 1)(1 + \lambda)(n + p)}. \]

\[
\cdot \sum_{k=p+1}^{\infty} k[1 + \lambda(k - p)]C(n, k, p)(|a_k| + |b_k|)r^{p+1} \geq \geq (1 - b_p)r^p - \frac{1}{(p + 1)(1 + \lambda)(n + p)} [p(1 - \alpha) - |b_p|] r^{p+1}. \]

The bounds given in Theorem 3.1 for the functions \( f = h + \overline{g} \) of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

\[
f(z) = z^p + b_p\overline{z}^p - \frac{p(1 - \alpha) - |b_p|}{(p + 1)(1 + \lambda)(n + p)} \overline{z}^{p+1}
\]

and

\[
f(z) = (1 - |b_p|)z^p - \frac{p(1 - \alpha) - |b_p|}{(p + 1)(1 + \lambda)(n + p)} z^{p+1}
\]

for \( |b_p| < 1 \) show that the bounds given Theorem 3.1 are sharp.
The following covering result follows from the second inequality in Theorem 3.1.

**Corollary 1** If \( f \in T_n^\lambda(p, \alpha) \), then

\[
\left\{ w : |w| < (1 - |b_p|) - \frac{p(1 - \alpha) - |b_p|}{(p + 1)(1 + \lambda)(n + p)} \right\} \subset f(\mathbb{U}).
\]

**Theorem 3.2.** \( f \in T_n^\lambda(p, \alpha) \) if and only if \( f \) can be expressed as

\[
(7) \quad f(z) = \sum_{k=p}^{\infty} (\gamma_k h_k + \mu_k g_k)
\]

where \( z \in \mathbb{U} \),

\[
h_p(z) = z^p, \quad h_k(z) = z^p - \frac{p(1 - \alpha)}{k[1 + \lambda(k - p)]C(n, k, p)} z^k, \quad (k = p + 1, p + 2, ...),
\]

\[
g_k(z) = z^p - \frac{p(1 - \alpha)}{k[1 + \lambda(k - p)]C(n, k, p)} z^k, \quad (k = p, p + 1, ...),
\]

\[
\sum_{k=p}^{\infty} (\gamma_k + \mu_k) = 1, \quad \gamma_k \geq 0 \text{ and } \mu_k \geq 0 (k = p + 1, p + 2, ...).
\]

In particular, the extreme points of \( T_n^\lambda(p, \alpha) \) are \( \{h_k\} \) and \( \{g_k\} \).

**Proof.** Note that for \( f \) we may write

\[
f(z) = \sum_{k=p}^{\infty} (\gamma_k h_k + \mu_k g_k) = \]

\[
= \sum_{k=p}^{\infty} (\gamma_k + \mu_k) z^p - \sum_{k=p+1}^{\infty} \frac{p(1 - \alpha)}{k[1 + \lambda(k - p)]C(n, k, p)} \gamma_k z^k - ...\]

\[-\sum_{k=p}^{\infty} \frac{p(1-\alpha)}{k[1+\lambda(k-p)]C(n,k,p)} \mu_k z^k = \]

\[= z^p - \sum_{k=p+1}^{\infty} \frac{p(1-\alpha)}{k[1+\lambda(k-p)]C(n,k,p)} \gamma_k z^k - \]

\[\phantom{=} - \sum_{k=p}^{\infty} \frac{p(1-\alpha)}{k[1+\lambda(k-p)]C(n,k,p)} \mu_k z^k \]

Then

\[\sum_{k=p+1}^{\infty} \left[ \frac{p(1-\alpha)}{k[1+\lambda(k-p)]C(n,k,p)} \gamma_k \right] \]

\[\phantom{=} - \sum_{k=p}^{\infty} \left[ \frac{p(1-\alpha)}{k[1+\lambda(k-p)]C(n,k,p)} \mu_k \right] \]

\[= p(1-\alpha) \left( \sum_{k=p}^{\infty} (\gamma_k + \mu_k) - \gamma_p \right) = p(1-\alpha)(1-\gamma_p) \leq p(1-\alpha) \]

and so \( f \in T_{\lambda}^\alpha(p,\alpha) \).

Conversely, suppose that \( f \in T_{\lambda}^\alpha(p,\alpha) \). Setting

\[\gamma_k = \frac{k[1+\lambda(k-p)]C(n,k,p)}{p(1-\alpha)} |a_k|(k = p + 1, p + 2, \ldots), \]

\[\mu_k = \frac{k[1+\lambda(k-p)]C(n,k,p)}{p(1-\alpha)} |b_k|(k = p, p + 1, p + 2, \ldots), \]

we obtain

\[f(z) = \sum_{k=p}^{\infty} (\gamma_k h_k + \mu_k g_k) \text{ as required.} \]

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References


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