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On a Subclass of p-valent Functions whose coefficients related to Beta Function [1]

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

In the present paper, by making use of the Beta function, we introduce a subclass $A_s^*(p, A, B, \alpha)$ of functions with negative and missing coefficients which are analytic and p-valent in the unit disc $U = \{ z : |z| < 1 \}$. We give basic properties for functions belonging to the class $A_s^*(p, A, B, \alpha)$ and obtain numerous sharp results in terms of the Beta function including coefficient estimate, distortion theorems, closure theorems, integral operators and linear combinations of several functions belonging to $A_s^*(p, A, B, \alpha)$. We also obtain radii of close-to-convexity, starlikeness and convexity for functions belonging to $A_s^*(p, A, B, \alpha)$. Furthermore, convolution properties of several functions belonging to the class $A_s^*(p, A, B, \alpha)$ are studied here. Various distortion inequalities for fractional calculus of functions in the $A_s^*(p, A, B, \alpha)$ are also given.

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1 Introduction

Let A_p $(p \geq 2)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to the class A_p is said to be in the class $A_p^*(p, A, B, \alpha)$ if and only if

$$Re\left\{\frac{f^{(p-1)}(z)}{p!z}\right\} > \frac{\alpha}{p}$$

for $-1 \leq B < A \leq 1$, $-1 \leq B < 0$, $0 \leq \alpha < p$ and all $z \in U$.

In the other words, $f(z) \in A_p^*(p, A, B, \alpha)$ if and only if there exists a function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that

$$\frac{f^{(p-1)}(z)}{p!z} = \left(1 - \frac{\alpha}{p}\right)\frac{1 + Aw(z)}{1 + Bw(z)} + \frac{\alpha}{p} \tag{1.2}$$

The condition (1.2) is equivalent to

$$\left|\frac{p\frac{f^{(p-1)}(z)}{p!z} - p}{[pB + (A - B)(p - \alpha)] - pB\frac{f^{(p-1)}(z)}{p!z}}\right| < 1, \quad z \in U. \tag{1.3}$$

Let A_s denote the subclass of A_p consisting of functions analytic and p-valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \tag{1.4}$$

where $a_{p+n} > 0, \quad k \geq 2$.

Let us define

$$A_s^*(p, A, B, \alpha) = A_p^*(p, A, B, \alpha) \cap A_s.$$

M.K. Aouf and H.E.Darwish [3], S.M.Sarangi and V.J.Patil [4] have studied certain classes of p-valent functions with negative and missing coefficients. M.K. Aouf and H.E.Darwish [2], S.L.Shukla and Dastrath [5]have studied certain classes of analytic functions with negative coefficients. Also the class A_p is studied by M.Nunokawa [1]. In this paper, while we were obtaining coefficient estimates, distortion theorem, covering theorem, integral operators, convolution properties and radii of close-to-convexity, starlikeness and convexity for functions belonging to $A_s^*(p, A, B, \alpha)$, we used the beta function. Further it is shown that this class is closed under "arithmetic mean and "convex linear combinations. Also distortion theorems for fractional calculus are shown.

2 Coefficient Estimates

Theorem 1 Let the function $f(z)$ defined by (1.4). Then $f(z) \in A_s^*(p, A, B, \alpha)$ if and only if

$$(2.1) \qquad (1 - B) \sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p, n+1)} a_{p+n} \leq (A - B)(p - \alpha).$$

where B denotes the beta function. The result is sharp.

Proof : Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\left| p\frac{f^{(p-1)}(z)}{p!z} - p \right| - \left| pB + (A - B)(p - \alpha) - pB\frac{f^{(p-1)}(z)}{p!z} \right|$$

$$= \left| -\sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} z^n \right| -$$

$$- \left| (A-B)(p-\alpha) + B \sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} z^n \right|$$

$$\leq (1-B) \sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} - (A-B)(p-\alpha) \quad ; \quad B<0$$

$$\leq 0$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f(z) \in A_s^*(p,A,B,\alpha)$. To prove the converse, assume that

$$\left| \frac{p\frac{f^{(p-1)}(z)}{p!z} - p}{pB + (A-B)(p-\alpha) - pB\frac{f^{(p-1)}(z)}{p!z}} \right| =$$

$$= \left| \frac{-\sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} z^n}{(A-B)(p-\alpha) + B\sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} z^n} \right| < 1.$$

Since $Re(z) \leq |z|$ for all z, we have

$$Re \left\{ \frac{\sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} z^n}{(A-B)(p-\alpha) + B\sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} z^n} \right\} < 1. \tag{2.2}$$

Choose values of z on the real axis so that $\frac{f^{(p-1)}(z)}{p!z}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$(1-B) \sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p,n+1)} a_{p+n} \leq (A-B)(p-\alpha) \tag{2.3}$$

which obviously is required assertion (2.1).

Finally, sharpness follows if we take

$$f(z) = z^p - \frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)} z^{p+n} (n \geq k, k \geq 2). \tag{2.4}$$

Corollary 1 Let the function $f(z)$ defined by (1.4). If $f(z) \in A_s^*(p, A, B, \alpha)$, then

$$a_{p+n} \leq \frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)}{(1-B)}. \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by (2.4).

3 Distortion Properties

Theorem 2 If the function $f(z)$ defined by (1.4) in the $A_s^*(p, A, B, \alpha)$ then for $|z| = r < 1$

$$r^p - \frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p, k+1)}{(1-B)} r^{p+k} \leq |f(z)| \leq r^p +$$

$$+\frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p, k+1)}{(1-B)} r^{p+k} \tag{3.1}$$

and

$$pr^{p-1} - \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p, k)}{(1-B)} r^{p+k-1} \leq |f'(z)| \leq pr^{p-1} +$$

$$+\frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p, k)}{(1-B)} r^{p+k-1} \tag{3.2}$$

All the inequalities are sharp.

Proof : Let $f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}$. From Theorem 1, we have

$$(1-B)\frac{1}{(k+1)\mathrm{B}(p, k+1)} \sum_{n=k}^{\infty} a_{p+n} \leq$$

$$\leq (1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}a_{p+n} \leq (A-B)(p-\alpha) \tag{3.3}$$

which immediately yields for $n \geq k$

$$\sum_{n=k}^{\infty} a_{p+n} \leq \frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{(1-B)} \tag{3.4}$$

and

$$\sum_{n=k}^{\infty}(p+n)a_{p+n} \leq \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p,k)}{(1-B)}. \tag{3.5}$$

Consequently, for $|z| = r < 1$, we obtain

$$|f(z)| \leq |z|^p + \sum_{n=k}^{\infty}|a_{p+n}||z|^{p+n} \leq r^p + r^{p+k}\sum_{n=k}^{\infty}a_{p+n} \leq r^p +$$

$$+\frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{(1-B)}r^{p+k}$$

and

$$|f(z)| \geq |z|^p - \sum_{n=k}^{\infty}|a_{p+n}||z|^{p+n} \geq r^p - r^{p+k}\sum_{n=k}^{\infty}a_{p+n} \geq r^p -$$

$$-\frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{(1-B)}r^{p+k}$$

which prove that the assertion (3.1) of Theorem 2.

Furthermore, for $|z| = r < 1$ and (3.5), we have

$$|f'(z)| \leq p|z|^{p-1} + \sum_{n=k}^{\infty}(p+n)|a_{p+n}||z|^{p+n-1} \leq pr^{p-1} + r^{p+k-1}\sum_{n=k}^{\infty}(p+n)a_{p+n}$$

$$\leq pr^{p-1} + \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p,k)}{(1-B)}r^{p+k-1}$$

and

$$|f'(z)| \geq p|z|^{p-1} - \sum_{n=k}^{\infty}(p+n)|a_{p+n}||z|^{p+n-1} \geq pr^{p-1} - r^{p+k-1}\sum_{n=k}^{\infty}(p+n)a_{p+n}$$

$$\geq pr^{p-1} - \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p,k)}{(1-B)} r^{p+k-1}$$

which prove that the assertion (3.2) of Theorem 2.

The bounds in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{(1-B)} z^{p+k} \quad ; z = \mp r. \tag{3.6}$$

Letting $r \to 1^-$ in the left hand side of (3.1), we have the following:

Corollary 2 If $f(z) \in A_s^*(p, A, B, \alpha)$, then the disc $|z| < 1$ is mapped by $f(z)$ onto a domain that contains the disc

$$|w| < \frac{(1-B) - (k+1)\mathrm{B}(p,k+1)(A-B)(p-\alpha)}{(1-B)}.$$

The result is sharp with the extremal function $f(z)$ being given by (3.6).

Putting $\alpha = 0$ in Theorem 2 and Corollary 2, we get

Corollary 3 If the function $f(z)$ defined by (1.4) in the $A_s^*(p, A, B, 0)$ then for $|z| = r$

$$r^p - \frac{(A-B)p(k+1)\mathrm{B}(p,k+1)}{(1-B)} r^{p+k} \leq |f(z)| \leq r^p +$$

$$+ \frac{(A-B)p(k+1)\mathrm{B}(p,k+1)}{(1-B)} r^{p+k}$$

and

$$pr^{p-1} - \frac{(A-B)pk(k+1)\mathrm{B}(p,k)}{(1-B)} r^{p+k-1} \leq |f'(z)| \leq pr^{p-1} +$$

$$+ \frac{(A-B)pk(k+1)\mathrm{B}(p,k)}{(1-B)} r^{p+k-1}.$$

The result is sharp with the extremal function

$$f(z) = z^p - \frac{(A-B)p(k+1)\mathrm{B}(p,k+1)}{(1-B)} z^{p+k} \quad ; z = \mp r. \tag{3.7}$$

Corollary 4 If $f(z) \in A_s^*(p, A, B, \alpha)$, then the disc $|z| < 1$ is mapped by $f(z)$ onto a domain that contains the disc

$$|w| < \frac{(1-B) - p(k+1)\mathrm{B}(p, k+1)(A-B)}{(1-B)}.$$

The result is sharp with the extremal function $f(z)$ being given by (3.7).

4 Radii Of Close-To-Convexity, Starlikeness And Convexity

Theorem 3 : Let the function $f(z)$ defined by (1.4) in the class $A_s^*(p, A, B, \alpha)$. Then $f(z)$ is p-valent close-to-convex of order δ $(0 \leq \delta < p)$ in $|z| < R_1$, where

$$(4.1) \qquad R_1 = \inf_{n \geq 2} \left\{ \left[\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)} \left(\frac{p-\delta}{p+n} \right) \right]^{\frac{1}{n}} \right\}$$

Theorem 4 : Let the function $f(z)$ defined by (1.4) in the class $A_s^*(p, A, B, \alpha)$. Then $f(z)$ is p-valent starlike of order δ $(0 \leq \delta < p)$ in $|z| < R_2$, where

$$(4.2) \quad R_2 = \inf_{n \geq 2} \left\{ \left[\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}.$$

Theorem 5 : Let the function $f(z)$ defined by (1.4) in the class $A_s^*(p, A, B, \alpha)$. Then $f(z)$ is p-valent convex function of order δ $(0 \leq \delta < p)$ in $|z| < R_3$, where

(4.3)

$$R_3 = \inf_{n \geq 2} \left\{ \left[\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)} \left(\frac{p(p-\delta)}{(p+n)(p+n-\delta)} \right) \right]^{\frac{1}{n}} \right\}.$$

The results in Theorem 3,4,5 are sharp with the extremal function $f(z)$ given by (2.4). Furthermore, taking $\delta = 0$ in Theorem 3,4,5, we obtain radius of close-to-convexity , starlikeness and convexity, respectively.

5 Integral Operators

Theorem 6 Let c be a real number such that $c > -p$. If $f(z) \in A_s^*(p, A, B, \alpha)$, then the function $F(z)$ defined by

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \tag{5.1}$$

also belongs to $A_s^*(p, A, B, \alpha)$.

Proof : Let $f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}$. Then from representation of $F(z)$, it follows that $F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}$ where $b_{p+n} = \left(\frac{c+p}{c+p+n}\right) a_{p+n}$. Therefore using Theorem 1 for the coefficients of $F(z)$ we have

$$(1-B) \sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p, n+1)} b_{p+n} =$$

$$= (1-B) \sum_{n=k}^{\infty} \frac{1}{(n+1)\mathrm{B}(p, n+1)} \left(\frac{c+p}{c+p+n}\right) a_{p+n} \leq (A-B)(p-\alpha)$$

since $\frac{c+p}{c+p+n} < 1$ and $f(z) \in A_s^*(p, A, B, \alpha)$. Hence $F(z) \in A_s^*(p, A, B, \alpha)$.

Theorem 7 Let c be a real number such that $c > -p$. If $F(z) \in A_s^*(p, A, B, \alpha)$, then the function $f(z)$ defined by (5.1) is p-valent in $|z| < R^*$, where

(5.2)

$$R^* = \inf_{n \geq 2} \left\{ \left(\frac{c+p}{c+p+n}\right) \left[\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)} \left(\frac{p}{p+n}\right)\right]^{\frac{1}{n}} \right\}.$$

The result is sharp.

Proof : Let $F(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}$. It follows from (5.1)

$$f(z) = \frac{z^{1-c}}{c+p}\frac{d}{dz}\left[z^c F(z)\right] = z^p - \sum_{n=k}^{\infty}\left(\frac{c+p+n}{c+p}\right) a_{p+n} z^{p+n}.$$

In order to obtain the required result it sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p$ for $|z| < R^*$ where R^* is defined by (5.2). Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{n=k}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right) a_{p+n} |z|^n .$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p$ if

$$\sum_{n=k}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right) a_{p+n} |z|^n < p. \tag{5.3}$$

But Theorem 1 confirms that

$$\sum_{n=k}^{\infty} p\left[\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\,\mathrm{B}(p,n+1)}\right] a_{p+n} \leq p.$$

Hence (5.3) will be satisfied if

$$(p+n)\left(\frac{c+p+n}{c+p}\right) a_{p+n} |z|^n \leq p\left[\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}\right] a_{p+n}$$

or if

$$|z| \leq \left\{\left(\frac{c+p}{c+p+n}\right)\frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}\left(\frac{p}{p+n}\right)\right\}^{\frac{1}{n}}.$$

Therefore $f(z)$ is p-valent in $|z| < R^*$.

Sharpness follows if we take

$$f(z) = z^p - \left(\frac{c+p+n}{c+p}\right)\frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)} z^{p+n}.$$

6 Closure Properties

In this section we show that the class $A_s^*(p,A,B,\alpha)$ is closed under "arithmetic mean and "convex linear combinations.

Theorem 8 Let $f_j(z)=z^p-\sum_{n=k}^{\infty}a_{p+n,j}z^{p+n}\quad j=1,2,...$ and $h(z)=z^p-\sum_{n=k}^{\infty}b_{p+n}z^{p+n}$, where $b_{p+n}=\sum_{j=1}^{\infty}\lambda_j a_{p+n,j}$, $\lambda_j>0$ and $\sum_{j=1}^{\infty}\lambda_j=1$. If $f_j(z)\in A_s^*(p,A,B,\alpha)$ for each $j=1,2,...$, then $h(z)\in A_s^*(p,A,B,\alpha)$.

Proof : If $f_j(z)\in A_s^*(p,A,B,\alpha)$, then we have from Theorem 1 that

$$(1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}a_{p+n,j}\le(A-B)(p-\alpha)\quad j=1,2,....$$

Therefore

$$(1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}b_{p+n}=$$

$$=(1-B)\sum_{n=k}^{\infty}\left[\frac{1}{(n+1)\mathrm{B}(p,n+1)}\left(\sum_{j=1}^{\infty}\lambda_j a_{p+n,j}\right)\right]\le(A-B)(p-\alpha).$$

Hence, by Theorem 1, $h(z)\in A_s^*(p,A,B,\alpha)$.

Theorem 9 The class $A_s^*(p,A,B,\alpha)$ is closed under convex linear combinations.

Proof : Let $f(z)=z^p-\sum_{n=k}^{\infty}a_{p+n}z^{p+n}$ and $g(z)=z^p-\sum_{n=k}^{\infty}b_{p+n}z^{p+n}$ $(k\ge p,\ a_{p+n}>0,\ b_{p+n}>0)$, be any two functions of the class $A_s^*(p,A,B,\alpha)$. For $\lambda\,(0\le\lambda\le1)$, it is sufficient to show that $h(z)=(1-\lambda)\,f(z)+\lambda g(z)$, $z\in\mathbb{U}$ is also a function of $A_s^*(p,A,B,\alpha)$. Now,

$$h(z)=z^p-\sum_{n=k}^{\infty}\left[(1-\lambda)\,a_{p+n}+\lambda b_{p+n}\right]z^{p+n}.$$

Applying Theorem 1 to $f,g\in A_s^*(p,A,B,\alpha)$, we have

$$(1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}\left[(1-\lambda)\,a_{p+n}+\lambda b_{p+n}\right]=$$

$$= (1-\lambda)(1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}a_{p+n}+$$

$$+\lambda(1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}b_{p+n}\leq$$

$$\leq (1-\lambda)(A-B)(p-\alpha)+\lambda(A-B)(p-\alpha)=(A-B)(p-\alpha).$$

Then $h(z)\in A_s^*(p,A,B,\alpha)$.

Theorem 10 Let $f_p(z)=z^p$ and $f_{p+n}(z)=z^p-\frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)}z^{p+n}$ $(n\geq k, k\geq 2)$. Then $f(z)\in A_s^*(p,A,B,\alpha)$ if and only if it can be expressed in the form

$$f(z)=\lambda_p f_p(z)+\sum_{n=k}^{\infty}\lambda_n f_{p+n}(z),\quad z\in\mathbb{U}$$

where $\lambda_n\geq 0$ and $\lambda_p=1-\sum_{n=k}^{\infty}\lambda_n$.

Proof : Let us assume that

$$f(z)=\lambda_p f_p(z)+\sum_{n=k}^{\infty}\lambda_n f_{p+n}(z)$$

$$=[1-\sum_{n=k}^{\infty}\lambda_n]z^p+\sum_{n=k}^{\infty}\lambda_n\left\{z^p-\frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)}z^{p+n}\right\}$$

$$=z^p-\sum_{n=k}^{\infty}\frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)}\lambda_n z^{p+n}.$$

Then from Theorem 1 we have

$$(1-B)\sum_{n=k}^{\infty}\frac{1}{(n+1)\mathrm{B}(p,n+1)}\frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)}\lambda_n$$

$$=(A-B)(p-\alpha)\sum_{n=k}^{\infty}\lambda_n\leq (A-B)(p-\alpha).$$

Hence $f(z)\in A_s^*(p,A,B,\alpha)$.

Conversely, let $f(z) \in A_s^*(p, A, B, \alpha)$. It follows from Corollary 1 that

$$a_{p+n} \leq \frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)}{(1-B)}.$$

Setting

$$\lambda_n = \frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)} a_{p+n}, \qquad n = k, k+1, \dots, k \geq 2$$

and $\lambda_p = 1 - \sum_{n=k}^{\infty} \lambda_n$, we have

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}$$

$$= z^p - \sum_{n=k}^{\infty} \lambda_n z^p + \sum_{n=k}^{\infty} \lambda_n z^p - \sum_{n=k}^{\infty} \lambda_n \frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)}{(1-B)} z^{p+n}$$

$$= [1 - \sum_{n=k}^{\infty} \lambda_n] z^p + \sum_{n=k}^{\infty} \lambda_n \left\{ z^p - \frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p, n+1)}{(1-B)} z^{p+n} \right\}$$

$$= \lambda_p f_p(z) + \sum_{n=k}^{\infty} \lambda_n f_{p+n}(z).$$

This completes the proof of Theorem 10.

7 Convolution Properties

Theorem 11 If $f_1(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}$ and $f_2(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}$ are in the class $A_s^*(p, A, B, \alpha)$ then $(f_1 * f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} b_{p+n} z^{p+n}$ is in the class $A_s^*(p, A, B, \psi)$, where

$$\psi = p - \frac{3(A-B)(p-\alpha)^2 \mathrm{B}(p, 3)}{(1-B)}.$$

The result is best possible for $f_1(z)$ and $f_2(z)$ given by

$$f_j(z) = z^p - \frac{3(A-B)(p-\alpha)^2 \mathrm{B}(p, 3)}{(1-B)} z^{p+2} \qquad j = 1, 2.$$

Proof : In order to prove our theorem, we have to find the largest $\psi = \psi(p, A, B, \alpha)$ such that

$$\sum_{n=k}^{\infty} \frac{(1-B)}{(A-B)(p-\psi)(n+1)\mathrm{B}(p,n+1)} a_{p+n} b_{p+n} \leq 1$$

for $f_1(z)$ and $f_2(z)$ in the class $A_s^*(p, A, B, \alpha)$. Since $f_1(z)$ and $f_2(z)$ are in the class $A_s^*(p, A, B, \alpha)$, in view of Theorem 1,

$$\sum_{n=k}^{\infty} \frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)} a_{p+n} \leq 1$$

and

$$\sum_{n=k}^{\infty} \frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)} b_{p+n} \leq 1.$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=k}^{\infty} \frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)} \sqrt{a_{p+n} b_{p+n}} \leq 1 \tag{7.1}$$

Thus it is sufficient to show that

$$\frac{(1-B)}{(A-B)(p-\psi)(n+1)\mathrm{B}(p,n+1)} a_{p+n} b_{p+n} \leq$$

$$\leq \frac{(1-B)}{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)} \sqrt{a_{p+n} b_{p+n}}$$

or

$$\sqrt{a_{p+n} b_{p+n}} \leq \frac{p-\psi}{p-\alpha}.$$

Note that

$$\sqrt{a_{p+n} b_{p+n}} \leq \frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)}.$$

Hence, we need only to prove that

$$\frac{(A-B)(p-\alpha)(n+1)\mathrm{B}(p,n+1)}{(1-B)} \leq \frac{p-\psi}{p-\alpha} \tag{7.2}$$

or, equivalently, that

$$\psi \leq p - \frac{(A-B)(p-\alpha)^2(n+1)\mathrm{B}(p,n+1)}{(1-B)}.$$

Defining the function $\Xi(n)$ by

$$\Xi(n) = p - \frac{(A-B)(p-\alpha)^2(n+1)\mathrm{B}(p,n+1)}{(1-B)}, \tag{7.3}$$

we see that $\Xi(n)$ is an increasing function of n. Therefore, letting $n = 2$ in (7.3), we obtain

$$\psi \leq \Xi(2) = p - \frac{3(A-B)(p-\alpha)^2\mathrm{B}(p,3)}{(1-B)}$$

which completes the assertion of theorem.

8 Definitions And Applications Of The Fractional Calculus

In this section, we shall prove several distortion theorems in terms of the beta function for functions to general class $A_s^*(p,A,B,\alpha)$. Each of these theorems would involve certain operators of fractional calculus we find it to be convenient to recall here the following definition which were used recently by Owa [6] (and more recently, by Owa & Srivastava [7], and Srivastava & Owa [8], ; see also Srivastava et all. [9])

Definition 1 The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0) \tag{8.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2 The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \quad (0\leq\lambda<1) \tag{8.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.

Definition 3 Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda} f(z)=\frac{d^n}{dz^n}D_z^{\lambda} f(z) \tag{8.3}$$

where $0\leq\lambda<1$ and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

From Definition 2, we have

$$D_z^0 f(z)=f(z) \tag{8.4}$$

which, in view of Definition 3 yields,

$$D_z^{n+0} f(z)=\frac{d^n}{dz^n}D_z^0 f(z)=f^n(z). \tag{8.5}$$

Thus, it follows from (8.4) and (8.5) that

$$\lim_{\lambda\to 0} D_z^{-\lambda} f(z)=f(z)$$

and

$$\lim_{\lambda\to 0} D_z^{1-\lambda} f(z)=f'(z).$$

Theorem 12 Let the function $f(z)$ defined by (1.4) be in the class $A_s^*(p, A, B, \alpha)$. Then for $z \in \mathbb{U}$ and $\lambda > 0$,

$$\left|D_z^{-\lambda} f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} |z|^{p+\lambda} \cdot$$

$$\cdot \left\{1 - \frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p+\lambda+1,k)\mathrm{B}(p,k+1)}{\mathrm{B}(p+1,k)(1-B)} |z|^k\right\}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} |z|^{p+\lambda} \cdot$$

$$\cdot \left\{1 + \frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p+\lambda+1,k)\mathrm{B}(p,k+1)}{\mathrm{B}(p+1,k)(1-B)} |z|^k\right\}.$$

The result is sharp.

Proof : Let

$$F(z) = \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) =$$

$$= z^p - \sum_{n=k}^{\infty} \frac{\mathrm{B}(p+\lambda+1,n)}{\mathrm{B}(p+1,n)} a_{p+n} z^{p+n}$$

$$= z^p - \sum_{n=k}^{\infty} \psi(n) a_{p+n} z^{p+n}$$

where

$$\psi(n) = \frac{\mathrm{B}(p+\lambda+1,n)}{\mathrm{B}(p+1,n)} \quad (n \geq k).$$

Since

$$0 < \psi(n) \leq \psi(k) = \frac{\mathrm{B}(p+\lambda+1,k)}{\mathrm{B}(p+1,k)},$$

we have, with the help of (3.4).

$$|F(z)| \geq |z|^p - \psi(k) |z|^{p+k} \sum_{n=k}^{\infty} a_{p+n} \geq$$

$$\geq |z|^p - \frac{\mathrm{B}(p+\lambda+1,k)(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{\mathrm{B}(p+1,k)(1-B)}\,|z|^{p+k}$$

and

$$|F(z)| \leq |z|^p + \psi(k)\,|z|^{p+k}\sum_{n=k}^{\infty} a_{p+n} \leq$$

$$\leq |z|^p + \frac{\mathrm{B}(p+\lambda+1,k)(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{\mathrm{B}(p+1,k)(1-B)}\,|z|^{p+k}$$

which prove the inequalities of Theorem 12. Further equalities are attained for the function

$$f(z) = z^p - \frac{(A-B)(p-\alpha)(k+1)\mathrm{B}(p,k+1)}{(1-B)} z^{p+k} \tag{8.6}$$

Theorem 13 Let the function $f(z)$ defined by (1.4) be in the class $A_s^*(p,A,B,\alpha)$. Then for $0 \leq \lambda < 1$,

$$\left|D_z^\lambda f(z)\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,|z|^{p-\lambda}\cdot$$

$$\cdot\left\{1 - \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p-\lambda+1,k)\mathrm{B}(p,1)}{(1-B)}\,|z|^k\right\}$$

and

$$\left|D_z^\lambda f(z)\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\,|z|^{p-\lambda}\cdot$$

$$\cdot\left\{1 + \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p-\lambda+1,k)\mathrm{B}(p,1)}{(1-B)}\,|z|^k\right\}.$$

The result is sharp for the function $f(z)$ given by (8.6).

The proof of Theorem 13 is obtained by using the same technique as in the proof of Theorem 12. Setting $\lambda = 0$ in Theorem 13, we obtain the following Corollary:

Corollary 8 If $f(z) \in A_s^*(p,A,B,\alpha)$, then

$$|f(z)| \geq |z|^p\left\{1 - \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p+1,k)}{(1-B)p}\,|z|^k\right\}$$

and

$$|f(z)| \leq |z|^p \left\{ 1 + \frac{(A-B)(p-\alpha)k(k+1)\mathrm{B}(p+1,k)}{(1-B)p} |z|^k \right\}$$

for $k \geq 2$, $p \in \mathbb{N}$ and for all $z \in \mathbb{U}$.

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