Properties regarding the trace of a matrix

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

There exist in many collection of mathematics problems applications concerning the trace of a matrix (ex. ...). We understand by the trace of a matrix the sum of all elements that are on the matrix first diagonal. The aim of this article is to present some properties regarding the trace of a matrix.

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Throughout the paper if \( A \) is an \( n \times n \) matrix, we write \( \text{tr} \ A \) to denote the trace of \( A \) and \( \text{det} \ A \) for the determinant of \( A \). If \( A \) is positive definite we write \( A > 0 \).

Application 1. Let \( A \in M_2(\mathbb{C}) \) and \( n \in \mathbb{N}^* \) with \( A^n = I_2 \). Show that if \( A + \text{det} \ A \) is a real matrix, then \( \text{tr} \ A \) and \( \text{det} \ A \) are real numbers.

Proof. Let \( f(x) = X^n - \text{tr} \ A \cdot X + \text{det} \ A \) be the characteristic polyoma

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of the $A$ matrix and $\lambda_1$, $\lambda_2$ its roots. Because $\lambda_1^n = \lambda_2^n = 1$, results that $|\lambda_1| = |\lambda_2| = 1$. By the hypothesis $\text{tr}A + \text{det}A = \lambda_1 + \lambda_2 + \lambda_1\lambda_2$ is a real number, so $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_1\lambda_2}$ from $\mathbb{R}$, so $\frac{\text{tr}A + 1}{\text{det}A}$ is real number. Then $1 + \frac{\text{tr}A + 1}{\text{det}A} = \frac{\text{tr}A + \text{det}A + 1}{\text{det}A}$ is real number and $\text{tr}A + \text{det}A + 1$ also, so $\text{det}A$ and then $\text{tr}A$ is real number.

**Application 2.** If $A > 0$ and $B > 0$, then
\[ 0 < \text{tr} (AB)^m \leq (\text{tr} (AB))^m \quad \text{for all} \quad m \in \mathbb{N}^* \]

**Proof.** The equality takes place for $n = 1$. If $n > 1$, for $B = I$ the inequality is true because $0 < \text{tr} (A^n) \leq (\text{tr} A)^n$, become
\[ \sum_{i=1}^{n} \lambda_i^m \leq \left( \sum_{i=1}^{n} \lambda_i \right)^m, \]
where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$.

If $A \mapsto AB$, the result has been proved.

**Application 3.** If $A_i > 0$ and $B_i > 0$ ($i = 1, 2, \ldots, k$) then
\[ \left( \text{tr} \sum_{i=1}^{k} A_iB_i \right)^n \leq \left( \text{tr} \sum_{i=1}^{k} A_i^n \right) \left( \text{tr} \sum_{i=1}^{k} B_i^n \right) \]

If $A_iB_i > 0$ ($i = 1, 2, \ldots, k$), then
\[ \left( \text{tr} \sum_{i=1}^{k} A_iB_i \right)^n \leq \left( \text{tr} \sum_{i=1}^{k} A_i^n \right) \left( \text{tr} \sum_{i=1}^{k} B_i^n \right) \]

**Proof.** Because
\[ 0 \leq \text{tr} \left( \sum_{i=1}^{k} \alpha A_i + B_i \right)^n = \alpha^n \text{tr} \left( \sum_{i=1}^{k} A_i^n \right) + \\
+2\alpha \text{tr} \left( \text{tr} \sum_{i=1}^{k} A_iB_i \right) + \text{tr} \left( \text{tr} \sum_{i=1}^{k} B_i^n \right) \]
the result has been proved.
In order to demonstrate the second inequality it is sufficient to demonstrate that

\[ \text{tr} \left( \sum_{i=1}^{k} A_i B_i \right)^n \leq \left( \text{tr} \sum_{i=1}^{k} A_i B_i \right) \]

Because \( A_i B_i > 0 \) for all \( i = 1, 2, \ldots, k \), we have \( U = \sum_{i=1}^{k} A_i B_i > 0 \). The inequality (1) will result by the fact that \( \text{tr} \left( U^n \right) \leq (\text{tr} U)^n \), for all \( U > 0 \).

**Application 4.** If \( A > 0 \) and \( B > 0 \) then

\[ n(\det A \det B)^{\frac mn} \leq \text{tr} \left( A^n B^n \right) \]

for any positive integer \( m \).

**Proof.** Since \( A \) is diagonalizable, there exists an orthodiagonal matrix \( P \) and a diagonal matrix \( D \) such that \( D = P^\top A \) (see [2]). So if the eigenvalues of \( A \) are \( \lambda_1, \lambda_2, \ldots, \lambda_n \), then \( D = \text{diag} \left( \lambda_1, \lambda_2, \ldots, \lambda_n \right) \).

Let \( b_{11}(m), b_{22}(m), \ldots, b_{nn}(m) \) denote the elements of \((PB^P)^n\). Then

\[
\frac{1}{n} \text{tr} \left( A^n B^n \right) = \frac{1}{n} \text{tr} \left( PD^n P^\top B^n \right) = \frac{1}{n} \text{tr} \left( D^n P^\top B^n P \right) = \]

\[
= \frac{1}{n} \text{tr} \left[ D^n (P^\top BP)^n \right] = \frac{1}{n} [\lambda_1^m b_{11}(m) + \lambda_2^m b_{22}(m) + \ldots + \lambda_n^m b_{nn}(m)].
\]

Using the arithmetic - mean geometric - mean inequality, we get

\[ \frac{1}{n} \text{tr} \left( A^n B^n \right) \leq [\lambda_1^m \lambda_2^m \ldots \lambda_n^m]^\frac{1}{n} [b_{11}(m) b_{22}(m) \ldots b_{nn}(m)]^{\frac{1}{n}}. \]

Since \( \det A \leq a_{11}a_{22} \ldots a_{nn} \) for any positive definite matrix \( A \), we conclude that

\[ \det \left( P^\top BP \right)^n \leq b_{11}(m)b_{22}(m) \ldots b_{nn}(m) \]

and

\[ \det D^n \leq \lambda_1^m \lambda_2^m \ldots \lambda_n^m. \]
Therefore from (2) it follows that
\[
\frac{1}{n} \text{tr} (A^n B^n) \leq [\det (D^n)]^{\frac{1}{n}} [\det (P^T BP)^m]^{\frac{1}{n}} =
\]
\[
= [\det (P^T AP)]^{\frac{m}{n}} [\det (P^T BP)]^{\frac{m}{n}} = (\det A \det B)^{\frac{m}{n}}.
\]
Here we used the fact that $A > 0$ and $B > 0$. The proof is complete.

**Corollary 1.** Let $A$ and $X$ be positive definite $n \times n$ - matrices such that $\det X = 1$. Then
\[
n(\det A)^{\frac{1}{n}} \leq \text{tr} (AX).
\]

**Proof.** Take $B = X$ and $m = 1$ in Applications 4.

**References**


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