On the means of sequences

Ioan Tîncu

Abstract
In this paper we investigate the invariancy of a class of real sequences with respect to the transformation \( A : a \rightarrow A(a) \).

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We consider the set of real sequences \( K \), the set \( K_m \) of all sequences which are convex of order \( m \) (\( m \in \mathbb{N} \)) and the operator \( \Delta^r : K \rightarrow \mathbb{R} \), \( r \in \mathbb{R} \), defined by

\[
\Delta^r a_n = (-1)^{|r|} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k},
\]

with the convention \( \Delta^0 a_n = a_n \) for every \( n \in \mathbb{N} \), where:

\[
(x)_l = x(x+1)...(x+l-1), l \in \mathbb{N}, (x)_0 = 1
\]

\([r]\)-represent integer part of the real number \( r \).

Definition 1.1. We say that a real sequence \( (a_n)_{n=1}^{\infty} \) is of \( M_r \) class if and only if

\[
\Delta^r a_n \geq 0 \text{ for every } n \in \mathbb{N} \text{ (see [5]).}
\]
If \( r \in \mathbb{N} \) then \( M_r = K_r \left( \Delta^r a_n = \sum_{k=0}^{r} (-1)^{r-k} \frac{r!}{k!} a_{n+k} \right) \)

**Property.** For real numbers \( r, r_1, r_2 \) we have:

i) \( \Delta^{r+1} a_n = \Delta^r a_{n+1} - \Delta^r a_n \), for every \( n \in \mathbb{N} \)

ii) \( \Delta^{r_1+r_2} a_n = \Delta^{r_1} (\Delta^{r_2} a_n) = \Delta^{r_2} (\Delta^{r_1} a_n) \) (see [5])

Let \( A(a) = (A_n(a))_{n=1}^{\infty} \), be the sequence of the means, that is

\[ A_n(a) = \frac{1}{n+1} \sum_{k=0}^{n} a_k, \ n = 0, 1, 2, \ldots \]

If \( S \) denotes a certain class of real sequences, then \( a_n \) interesting problem is to investigate if this class is invariant with respect to the transformation \( A : a \rightarrow A(a) \); in other words if \( A(S) \subseteq S \). For instance, it is well-known that \( A(S_0) \subseteq S_0 \), \( S_0 \) being the class of all real sequences which are convergent.

In [1]-[4] it is shown that from the \( n \)-th order convexity of \( a = (a_n) \) follows the convexity, of the same order, of the sequence \( A(a) = (A_n(a)) \), i.e. \( A(K_m) \subseteq K_m \).

We shall find a representation of \( \Delta^r A_n(a) \) as a positive linear combinations of \( \Delta^r a_0, \Delta^r a_1, \ldots, \Delta^r a_n \).

**Theorem 1.1.** For \( r \geq 0 \) and \( n = 0, 1, 2, \ldots \) the equality

\[ \Delta^r A_n(a) = \sum_{k=0}^{n} c_k(n, r), \Delta^r a_k \] with

\[ c_k(n, r) = \begin{cases} \frac{n!}{(r+1)^{n+1}}, & k = 0 \\ \frac{n!}{(r+2)^n} \cdot \frac{(r+2)k-1}{k!}, & k = 1, 2, \ldots n. \end{cases} \] is verified.

**Proof.** For \( k = 0, 1, 2, \ldots \) we have:

\[ a_k = (k+1)A_k(a) - kA_{k-1}(a) \]

\[ \Delta^r a_k = (-1)^r \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} a_{k+i} = \]

\[ = (-1)^r \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} [(k+i+1)A_{k+i}(a) - (k+i)A_{k+i-1}(a)] = \]

\[ = (-1)^r \left[ \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} (k+i+1)A_{k+1}(a) - \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} (k+1)A_{k+1-1}(a) \right] = \]
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\[ (-1)^r \left\{ \sum_{i=0}^{\infty} \frac{(-r) i}{i!} (k + i + 1)A_{k+i}(a) - kA_{k-1}(a) \right\} = \]

\[ = (-1)^r \left\{ \sum_{i=0}^{\infty} \frac{(-r) i}{i!} (k + i + 1)A_{k+i}(a) - \right\} \]

\[ = (-1)^r \left\{ \sum_{i=0}^{\infty} \frac{(-r) i}{i!} (k + i + 1)A_{k+i}(a) - kA_{k-1}(a) \right\} = \]

\[ = (1 + r) \Delta^r A_k(a) - k(-1)^r \left[ \sum_{i=0}^{\infty} \frac{(-r - 1) i}{i!} A_{k+i}(a) - A_{k-1}(a) \right] = \]

\[ = (1 + r) \sum_{i=0}^{\infty} \frac{(-r) i}{i!} A_{k+i}(a) = \]

\[ = (1 + r) \Delta^r A_k(a) - k(-1)^r \sum_{i=0}^{\infty} \frac{(-r - 1) i}{i!} A_{k+i} = \]

\[ = (1 + r) \Delta^r A_k(a) + k \Delta^{r+1} A_{k-1}(a). \]

From property ii), \( \Delta^{r+1} A_{k-1}(a) = \Delta^r A_k(a) - \Delta^r A_{k-1}(a) \).

We obtain

\[ \Delta^r a_k = (1 + r + k) \Delta^r A_k(a) - k \Delta^r A_{k-1}(a), \]

\[ \frac{(r + 2) k - 1}{k!} \Delta^r a_k = \frac{(r + 2) k}{k!} \Delta^r A_k(a) - \frac{(r + 2) k - 1}{(k - 1)!} \Delta^r A_{k-1}(a). \]

By summing these equalities we obtain

\[ \sum_{k=1}^{n} \frac{(r + 2) k - 1}{k!} \Delta^r a_k = \frac{(r + 2) n}{n!} \Delta^r A_n(a) - \Delta^r A_0(a). \]

In virtue of (5), for \( k = 0, \Delta^r A_0(a) = \frac{1}{r + 1} \Delta^r a_0. \)
We obtain
\[ \Delta^r A_n(a) = \frac{1}{r+1} \cdot \frac{n!}{(r+2)^n} \Delta^r a_0 + \frac{n!}{(r+2)^n} \sum_{k=1}^{n} \frac{(r+2)_{k-1}}{k!} \Delta^r a_k, \]
\[ \Delta^r A_n(a) = \frac{n!}{(r+1)^{n+1}} \Delta^r a_0 + \frac{n!}{(r+2)^n} \sum_{k=1}^{n} \frac{(r+2)_{k-1}}{k!} \Delta^r a_k. \]

**Theorem 1.2.** Let \( a = (a_n), A(a) = (A(a_n)) \); then:

i) \( A_n(M_r) \subseteq M_r \)

ii) if there exists \( C \in \mathbb{R} \), such that
\[ |\Delta^r (a_n)| < C, \ n = 0, 1, 2, \ldots \]

then for \( n = 1, 2, \ldots \)
\[ |\Delta^r A_n(a)| < \frac{C}{r+1} \]

**Proof.** The assertions i), ii) are consequences of the equalities (3) and (4).

**References**


"Lucian Blaga” University of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, no. 5–7
550012 Sibiu - Romania
E-mail address: tincu_ioan@yahoo.com