Integral Means and fractional calculus operators for comprehensive family of univalent functions with negative coefficients

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Abstract
In this paper, we obtain the integral means inequality for the function $f(z)$ belongs to the class $UT(\Phi, \Psi, \gamma, k)$ of analytic and univalent functions with negative coefficients defined in [3] with the extremal functions of this class. And also we derive some distortion theorems using fractional calculus techniques for the class $UT(\Phi, \Psi, \gamma, k)$.

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1 Introduction and definitions

Let $A$ denote the class of functions of the form

\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

which are analytic and univalent in the open disc $\mathcal{U} = \{ z : z \in \mathbb{C}, |z| < 1 \}$.

Also denote by $T$ the subclass of $A$ consisting of functions of the form

\begin{equation}
    f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad z \in \mathcal{U}
\end{equation}

introduced and studied by Silverman [16].

Following Goodman [5, 6], Rønning [12, 13] introduced and studied the following subclasses

(i) A function $f \in A$ is said to be in the class $S_p(\gamma, k)$, $k$–uniformly starlike functions of order $\gamma$, if it satisfies the condition

\begin{equation}
    \Re \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > k \left| \frac{zf''(z)}{f(z)} - 1 \right|, \quad z \in \mathcal{U}, \quad 0 \leq \gamma < 1 \text{ and } k \geq 0.
\end{equation}

(ii) A function $f \in A$ is said to be in the class $UCV(\gamma, k)$, $k$–uniformly convex functions of order $\gamma$, if it satisfies the condition

\begin{equation}
    \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} - \gamma \right\} > k \left| \frac{zf'''(z)}{f'(z)} \right|, \quad z \in \mathcal{U}, \quad 0 \leq \gamma < 1 \text{ and } k \geq 0.
\end{equation}

Indeed it follows from (1.3) and (1.4) that

\begin{equation}
    f \in UCV(\gamma, k) \iff zf' \in S_p(\gamma, k).
\end{equation}

**Definition 1.1 ([3]).** Given $\gamma(-1 \leq \gamma < 1), k(k \geq 0)$ and functions

\[ \Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \text{ and } \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \]
analytic in $U$, such that $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$ for $n \geq 2$, we let $f \in A$ is in $U(\Phi, \Psi, \alpha, \beta)$ if $(f \ast \Psi)(z) \neq 0$ and

$$\text{Re} \left\{ \frac{(f \ast \Phi)(z)}{(f \ast \Psi)(z)} - \gamma \right\} \geq k \left| \frac{(f \ast \Phi)(z)}{(f \ast \Psi)(z)} - 1 \right|, \quad \forall \ z \in U,$$

where (*) stands for the Hadamard product.

Further let $UT(\Phi, \Psi, \alpha, \beta) = U(\Phi, \Psi, \alpha, \beta) \cap T$.

We note that, by taking suitable choice of $\Phi$, $\Psi$, $\alpha$ and $\beta$ we obtain the following subclasses studied in literature.

1. $UT\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \gamma, 1\right) = TS_p(\gamma)$ (Subrmanian et al., [22])

2. $UT\left(\frac{z+2z^2}{(1-z)^2}, \frac{z}{1-z}, \gamma, k\right) = S_p T(\gamma, k)$ (Bharati et al., [1])

3. $UT\left(\frac{z+2z^2}{(1-z)^2}, \frac{z}{(1-z)^2}, 0, 1\right) = UCT$ (Subrmanian et al., [21])

4. $UT\left(\frac{z+2z^2}{(1-z)^2}, \frac{z}{(1-z)^2}, 0, k\right) = UCT(k)$ (Subrmanian et al., [21])

5. $UT\left(\frac{z+2z^2}{(1-z)^2}, \frac{z}{(1-z)^2}, \gamma, 1\right) = UCT(\gamma)$ (Bharati et al., [1])

6. $UT\left(\frac{z+2z^2}{(1-z)^2}, \frac{z}{(1-z)^2}, \gamma, k\right) = UCT(\gamma, k)$ (Bharati et al., [1])

7. $UT\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \gamma, 0\right) = S^*_T(\gamma)$ (Silverman [16])
8. \( UT\left(\frac{z+z^2}{(1-z^2)^2}, \gamma, 0\right) = K_T(\gamma) \) (Silverman [16])

9. \( UT(\Phi, \Psi, \gamma, 0) = E_T(\Phi, \Psi, \gamma) \) (Juneja et al.[7]).

10. \( UT(\Phi, \Psi, \frac{1+\beta-2\alpha}{2(1-\alpha)}, 0) = B_T(\Phi, \Psi, \alpha, \beta) \) (Frasin [4]).

In fact many subclasses of \( T \) are defined and studied to investigate coefficient estimates, extreme points, convolution properties and closure properties etc. suitably choosing \( \Phi, \Psi, \gamma \) and \( k \).

In this paper, we obtain integral means inequalities for functions \( f(z) \in UT(\Phi, \Psi, \gamma, k) \) and also we state integral means results for the classes studied in [21, 1, 22, 16, 4] as corollaries.

For analytic functions \( g(z) \) and \( h(z) \) with \( g(0) = h(0) \), \( g(z) \) is said to be subordinate to \( h(z) \) if there exists an analytic function \( w(z) \) so that \( w(0) = 0, |w(z)| < 1 \) (\( z \in \mathcal{U} \)) and \( g(z) = h(w(z)) \), we denote this subordination by \( g(z) \prec h(z) \).

To prove our main results, we need the following lemmas.

**Lemma 1.1 ([3])**. A function \( f(z) \in UT(\Phi, \Psi, \gamma, k) \) for \( \gamma(-1 \leq \gamma < 1) \) and \( k(k \geq 0) \) if and only if

\[
\sum_{n=2}^{\infty} [((1+k)\lambda_n - (\gamma + k)\mu_n] a_n \leq 1 - \gamma.
\]

The result is sharp with the extremal functions

\[
f_n(z) = z - \frac{1-\gamma}{\sigma(\gamma, k, n)} z^n, \quad n \geq 2
\]
where \( \sigma(\gamma, k, n) = (1 + k)\lambda_n - (\gamma + k)\mu_n, \gamma(-1 \leq \gamma < 1), k(k \geq 0) \) and \( n \geq 2 \).

**Lemma 1.2 ([8]).** If the functions \( f(z) \) and \( g(z) \) are analytic in \( U \) with \( g(z) \prec f(z) \) then

\[
(1.3) \quad \int_0^{2\pi} |g(re^{i\theta})| \eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})| \eta d\theta \quad \eta > 0, \quad z = re^{i\theta} \quad \text{and} \quad 0 < r < 1.
\]

## 2 Integral mean

Applying Lemma 1.1 and Lemma 1.2, we prove the following theorem.

**Theorem 2.1.** Let \( \eta > 0 \). If \( f(z) \in UT(\Phi, \Psi, \gamma, k), -1 \leq \gamma < 1, k \geq 0 \) and \( \{\sigma(\gamma, k, n)\}_{n=2}^{\infty} \) is non-decreasing sequence, then for \( z = re^{i\theta} \) and \( 0 < r < 1 \), we have

\[
(2.1) \quad \int_0^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})| \eta d\theta
\]

where \( f_2(z) = z - \frac{1 - \gamma}{\sigma(\gamma, k, 2)} z^2 \).

**Proof.** Let \( f(z) \) of the form (1.2) and \( f_2(z) = z - \frac{(1 - \gamma)}{\sigma(\gamma, k, 2)} z^2 \), then we must show that

\[
\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right| \eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1 - \gamma)}{\sigma(\gamma, k, 2)} z \right| \eta d\theta.
\]

By Lemma 1.2, it suffices to show that

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1 - \gamma}{\sigma(\gamma, k, 2)} z
\]
Setting

\begin{equation}
1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1 - \gamma}{\sigma(\gamma, k, 2)} w(z).
\end{equation}

From (2.2) and (1.1), we obtain

\[ |w(z)| = \left| \sum_{n=2}^{\infty} \frac{\sigma(\gamma, k, 2)}{1 - \gamma} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{\sigma(\gamma, k, n)}{1 - \gamma} a_n \leq |z| < 1. \]

This completes the proof of the Theorem 2.1.

By taking different choices of \( \Phi, \Psi, \gamma \) and \( k \) in the above theorem, we can state the following integral means results for various subclasses studied earlier [21, 1, 22, 16, 4].

**Corollary 2.2.** Let \( \eta > 0 \). If \( f(z) \in UT\left(\frac{z + z^2}{(1-z)^2}, \frac{z}{(1-z)^2}, 0, 1\right) = UCT \), then for \( z = re^{i\theta}; 0 < r < 1 \), we have

\begin{equation}
\int_{0}^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_{0}^{2\pi} |g_2(re^{i\theta})| \eta d\theta
\end{equation}

where \( g_2(z) = z - \frac{z^2}{6} \).

**Corollary 2.3.** Let \( \eta > 0 \). If \( f(z) \in UT\left(\frac{z + z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 0, k\right) = UCT(k) \) and \( k \geq 0 \), then for \( z = re^{i\theta}; 0 < r < 1 \), we have

\begin{equation}
\int_{0}^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_{0}^{2\pi} |g_2(re^{i\theta})| \eta d\theta
\end{equation}

where \( g_2(z) = z - \frac{z^2}{2(k+2)} \).
Corollary 2.4. Let $\eta > 0$. If $f(z) \in UT \left( \frac{z + z^2}{(1-z)^2}, \frac{z}{(1-z)^2}, \gamma, 1 \right) = UCT(\gamma)$ and $-1 \leq \gamma < 1$, then for $z = re^{i\theta}; 0 < r < 1$, we have

$$
\int_0^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_0^{2\pi} |g_2(re^{i\theta})| \eta d\theta
$$

where $g_2(z) = z - \frac{(1-\gamma)}{2(3-\gamma)} z^2$.

Corollary 2.5. Let $\eta > 0$. If $f(z) \in UT \left( \frac{z + z^2}{(1-z)^2}, \frac{z}{1-z}, \gamma, k \right) = UCT(\gamma, k)$, $-1 \leq \gamma < 1$ and $k \geq 0$, then for $z = re^{i\theta}; 0 < r < 1$, we have

$$
\int_0^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_0^{2\pi} |g_2(re^{i\theta})| \eta d\theta
$$

where $g_2(z) = z - \frac{(1-\gamma)}{2(2-\gamma+k)} z^2$.

Corollary 2.6. Let $\eta > 0$. If $f(z) \in UT \left( \frac{z}{1-z}, \frac{z}{1-z}, \gamma, 1 \right) = TS_p(\gamma)$ and $-1 \leq \gamma < 1$, then for $z = re^{i\theta}; 0 < r < 1$, we have

$$
\int_0^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_0^{2\pi} |g_2(re^{i\theta})| \eta d\theta
$$

where $g_2(z) = z - \frac{(1-\gamma)}{(3-\gamma)} z^2$.

Corollary 2.7. Let $\eta > 0$. If $f(z) \in UT \left( \frac{z}{1-z}, \frac{z}{1-z}, \gamma, k \right) = S_pT(\gamma, k)$, $-1 \leq \gamma < 1$ and $k \geq 0$, then for $z = re^{i\theta}; 0 < r < 1$, we have

$$
\int_0^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_0^{2\pi} |g_2(re^{i\theta})| \eta d\theta
$$

where $g_2(z) = z - \frac{(1-\gamma)}{(2-\gamma+k)} z^2$.

By taking $\gamma = \frac{1+\beta-2\alpha}{2(1-\alpha)}$ and $k = 0$ in Theorem 2.1 we get the following result of Frasin and Darus [4].
Corollary 2.8. Let $\eta > 0$. If $f(z) \in UT(\Phi, \Psi, \frac{1+\beta-2\alpha}{2(1-\alpha)}, 0) = B_T(\Phi, \Psi, \alpha, \beta)$, $0 \leq \beta < 1$ and $0 \leq \alpha < 1$, then for $z = re^{i\theta}$, $0 < r < 1$, we have

\[\int_{0}^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_{0}^{2\pi} |g_2(re^{i\theta})| \eta d\theta\]

where $g_2(z) = z - \frac{(1-\beta)}{\psi(\alpha, \beta, 2)} z^2$ and $\psi(\alpha, \beta, 2) = 2(1-\alpha)\lambda_2 - (1 + \beta - 2\alpha)\mu_2$.

Corollary 2.9. Let $\eta > 0$. If $f(z) \in UT\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \gamma, 0\right) = S_T^+(\gamma)$ and $\gamma \geq 0$, then for $z = re^{i\theta}$, $0 < r < 1$, we have

\[\int_{0}^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_{0}^{2\pi} |g_2(re^{i\theta})| \eta d\theta\]

where $g_2(z) = z - \frac{1-\gamma}{2-\gamma} z^2$.

Corollary 2.10. Let $\eta > 0$. If $f(z) \in UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \gamma, 0\right) = K_T(\gamma)$, and $\gamma \geq 0$, then for $z = re^{i\theta}$, $0 < r < 1$, we have

\[\int_{0}^{2\pi} |f(re^{i\theta})| \eta d\theta \leq \int_{0}^{2\pi} |g_2(re^{i\theta})| \eta d\theta\]

where $g_2(z) = z - \frac{1-\gamma}{2(2-\gamma)} z^2$.

Remark 2.11. If we take $\gamma = 0$ in $S_T^+(\gamma)$ of Corollary 2.9 and $K_T(\gamma)$ of Corollary 2.10, we get the integral means results obtained by Silverman [17].

3 Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [2],[9],[11], [14], [15], [18]and[19]). We find it to be convenient to
recall here the following definitions which are used earlier by Owa [10](and, subsequently, by Srivastava and Owa [19]).

**Definition 3.1.** The fractional integral of order $\xi$ is defined, for a function $f(z)$, by

\begin{equation}
D_{z}^{-\xi}f(z) = \frac{1}{\Gamma(\xi)} \int_{0}^{z} \frac{f(\zeta)}{(z - \zeta)^{1-\xi}} d\zeta \quad (\xi > 0),
\end{equation}

where the function $f(z)$ is analytic in a simply-connected region of the $z$-plane containing the origin and the multiplicity of the function $(z - \zeta)^{\xi-1}$ is removed by requiring the function $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

**Definition 3.2.** The fractional derivative of order $\xi$ is defined, for a function $f(z)$, by

\begin{equation}
D_{z}^{\xi}f(z) = \frac{1}{\Gamma(-\xi)} \frac{d}{dz} \int_{0}^{z} \frac{f(\zeta)}{(z - \zeta)^{1-\xi}} d\zeta \quad (0 \leq \xi < 1),
\end{equation}

where the function $f(z)$ is constrained, and the multiplicity of the function $(z - \zeta)^{-\xi}$ is removed as in Definition 3.1.

**Definition 3.3.** Under the hypotheses of Definition 3.2, the fractional derivative of order $n + \lambda$ is defined by

\begin{equation}
D_{z}^{m+\xi}f(z) = \frac{d^{m}}{dz^{m}} D_{z}^{\xi}f(z) \quad (0 \leq \xi < 1; \ m \in \mathbb{N}_0).
\end{equation}

**Remark 3.4.** From Definition 3.2, we have $D_{z}^{0}f(z) = f(z)$, which in view of Definition 3.3 yields $D_{z}^{m+0}f(z) = \frac{d^{m}}{dz^{m}} D_{z}^{0}f(z) = f^{(m)}(z)$. Thus, $\lim_{\xi \to 0} D_{z}^{-\xi}f(z) = f(z)$ and $\lim_{\xi \to 0} D_{z}^{1-\xi}f(z) = f'(z)$.

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa[20].
Definition 3.5 For real number $\eta > 0$, $\mu$ and $\delta$, the fractional integral operator $I_{0, z}^{\eta, \mu, \delta}$ is defined by

\begin{equation}
I_{0, z}^{\eta, \mu, \delta} f(z) = \frac{z^{-\eta - \mu}}{\Gamma(\eta)} \int_{0}^{z} (z - t)^{\eta - 1} F(\eta + \mu, -\delta; \eta; 1 - t/z) f(t) \, dt,
\end{equation}

where a function $f(z)$ is analytic in a simply-connected region of the $z$-plane containing the origin with the order

$$f(z) = O(|z| \varepsilon) \quad (z \to 0),$$

with $\varepsilon > \max\{0, \mu - \delta\} - 1$. Here $F(a, b; c; z)$ is the Gauss hypergeometric function defined by

\begin{equation}
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n,
\end{equation}

where $(\nu)_n$ is the Pochhammer symbol defined by

\begin{equation}
(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(\nu)} = \begin{cases} 
1 & (n = 0) \\
\nu(\nu + 1)(\nu + 2) \cdots (\nu + n - 1) & (n \in \mathbb{N})
\end{cases}
\end{equation}

and the multiplicity of $(z - t)^{\eta - 1}$ is removed by requiring $\log (z - t)$ to be real when $z - t > 0$.

Remark 3.4. For $\mu = -\eta$, we note that

$$I_{0, z}^{\eta, -\eta, \delta} f(z) = D^{-\eta}_{z} f(z).$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [20].
Lemma 3.7. If $\eta > 0$ and $n > \mu - \delta - 1$, then

\begin{equation}
I_{0,z}^{\eta,\mu,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\mu+\delta+1)}{\Gamma(n-\mu+1)\Gamma(n+\eta+\delta+1)} z^{n-\mu}.
\end{equation}

With aid of Lemma 3.7, we prove

Theorem 3.8. Let $\eta > 0$, $\mu < 2$, $\eta + \delta > -2$, $\mu - \delta < 2$, $\mu(\eta + \delta) \leq 3\eta$. Let the function $f(z)$ defined by (1.2) be in the class $UT(\Phi, \Psi, \gamma, k)$. If \( \{\sigma(\gamma, k, n)\}_{n=2}^{\infty} \) is a non-decreasing sequence. Then we have

\begin{equation}
\left| I_{0,z}^{\eta,\mu,\delta} f(z) \right| \geq \frac{\Gamma(2-\mu+\delta)}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)} \left| z \right|^{1-\mu} \left( 1 - \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)} \sigma(\gamma, k, 2) \left| z \right| \right)
\end{equation}

and

\begin{equation}
\left| I_{0,z}^{\eta,\mu,\delta} f(z) \right| \leq \frac{\Gamma(2-\mu+\delta)}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)} \left| z \right|^{1-\mu} \left( 1 + \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)} \sigma(\gamma, k, 2) \left| z \right| \right)
\end{equation}

for $z \in U_0$, where

\begin{equation}
U_0 = \begin{cases} 
U & (\mu \leq 1), \\
U - \{0\} & (\mu > 1).
\end{cases}
\end{equation}

The equalities in (3.8) and (3.9) are attained for the function $f(z)$ given by

\begin{equation}
f(z) = z - \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)} \sigma(\gamma, k, 2) z^2.
\end{equation}

Proof. By using Lemma 3.7, we have
$I_{0,z}^{\eta,\mu,\delta} f(z) = \frac{\Gamma(2-\mu+\delta)}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)} z^{1-\mu}$

(3.12) \[-\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\mu+\delta+1)}{\Gamma(n-\mu+1)\Gamma(n+\eta+\delta+1)} a_n z^{n-\mu} \quad (z \in U_0).\]

Letting

$G(z) = \frac{\Gamma(2-\mu)\Gamma(2+\eta+\delta)}{\Gamma(2-\mu+\delta)} z^{\mu I_{0,z}^{\eta,\mu,\delta} f(z)}$

(3.13) \[= z - \sum_{n=2}^{\infty} g(n) a_n z^n,\]

where

(3.14) \[g(n) = \frac{(2-\mu+\delta)_{n-1}(1)_n}{(2-\mu)_{n-1}(2+\eta+\delta)_{n-1}} \quad (n \geq 2),\]

we can see that the function $g(k)$ is non-increasing for integers $n(n \geq 2)$, and thus we have

(3.15) \[0 < g(n) \leq g(2) = \frac{2(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)}.\]

From Lemma 1.1, we obtain

(3.16) \[\sigma(\gamma, k, 2) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \sigma(\gamma, k, n) a_n \leq 1 - \gamma\]
Hence, using (3.15) and (3.16), we have

\[(3.17)\]

\[|G(z)| \geq |z| - g(2)|z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{2(1-\gamma)(2 - \mu + \delta)}{(2 - \mu)(2 + \eta + \delta)\sigma(\gamma, k, 2)} |z|^2,
\]

and

\[(3.18)\]

\[|G(z)| \leq |z| + g(2)|z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{2(1-\gamma)(2 - \mu + \delta)}{(2 - \mu)(2 + \eta + \delta)\sigma(\gamma, k, 2)} |z|^2,
\]

for \(z \in U_0\), where \(U_0\) is defined by (3.10). This completes the proof Theorem 3.8.

By using the same proof as in Theorem 3.8, we can prove

**Theorem 3.9.** Let the function \(f(z)\) be defined by (1.2) be in the class \(UT(\Phi, \Psi, \gamma, k)\). If \(\{\sigma(\gamma, k, n)/n\}_{n=2}^{\infty}\) is a non-decreasing sequence. Then we have

\[(3.19)\]

\[
\left|I_{0,z}^{\eta,\mu,\delta} f(z)\right| \geq \frac{\Gamma(2 - \mu + \delta)}{\Gamma(2 - \mu)\Gamma(2 + \eta + \delta)} \frac{|z|^{1-\mu}}{1 - \frac{4(1-\gamma)(2 - \mu + \delta)}{(2 - \mu)(2 + \eta + \delta)\sigma(\gamma, k, 2)} |z|},
\]

and

\[(3.20)\]

\[
\left|I_{0,z}^{\eta,\mu,\delta} f(z)\right| \leq \frac{\Gamma(2 - \mu + \delta)}{\Gamma(2 - \mu)\Gamma(2 + \eta + \delta)} \frac{|z|^{1-\mu}}{1 + \frac{4(1-\gamma)(2 - \mu + \delta)}{(2 - \mu)(2 + \eta + \delta)\sigma(\gamma, k, 2)} |z|},
\]

for \(z \in U_0\), where \(U_0\) is defined by (3.10). The equalities in (3.19) and (3.20) are attained for the function \(f(z)\) given by (3.11).

Taking \(\mu = -\eta = -\xi\) Theorem 3.8, we get
Corollary 3.10. Let the function $f(z)$ defined by (1.2) be in the class $U\Phi(\Phi, \Psi, \gamma, k)$. If $\{\sigma(\gamma, k, n)\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have

\begin{equation}
|D^{-\xi} f(z)| \geq \frac{|z|^{1+\xi}}{\Gamma(2 + \xi)} \left( 1 - \frac{2(1 - \gamma)}{(2 + \xi)\sigma(\gamma, k; 2)} |z| \right)
\end{equation}

and

\begin{equation}
|D^{-\xi} f(z)| \leq \frac{|z|^{1+\xi}}{\Gamma(2 + \xi)} \left( 1 + \frac{2(1 - \gamma)}{(2 + \xi)\sigma(\gamma, k; 2)} |z| \right)
\end{equation}

for $\xi > 0, z \in U$. The result is sharp for the function

\begin{equation}
D^{-\xi} f(z) = \frac{|z|^{1+\xi}}{\Gamma(2 + \xi)} \left( 1 - \frac{2(1 - \gamma)}{(2 + \xi)\sigma(\gamma, k; 2)} |z| \right)
\end{equation}

Taking $\mu = -\eta = \xi$ in Theorem 3.9, we get

Corollary 3.11. Let the function $f(z)$ defined by (1.2) be in the class $U\Phi(\Phi, \Psi, \gamma, k)$. If $\{\sigma(\gamma, k, n)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have

\begin{equation}
|D^{\xi} f(z)| \geq \frac{|z|^{1-\xi}}{\Gamma(2 - \xi)} \left( 1 - \frac{4(1 - \gamma)}{(2 - \xi)\sigma(\gamma, k; 2)} |z| \right)
\end{equation}

and

\begin{equation}
|D^{\xi} f(z)| \leq \frac{|z|^{1-\xi}}{\Gamma(2 - \xi)} \left( 1 + \frac{4(1 - \gamma)}{(2 - \xi)\sigma(\gamma, k; 2)} |z| \right)
\end{equation}

for $0 \leq \xi < 1, z \in U$. The result is sharp for the function given by (3.23).
Letting $\xi = 0$ in Corollary 3.10, we have

**Corollary 3.12 ([3]).** Let the function $f(z)$ defined by (1.2) be in the class $UT(\Phi, \Psi, \gamma, k)$. If $\{\sigma(\gamma, k, n)\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have

\begin{equation}
1 - \frac{1 - \gamma}{\sigma(\gamma, k, 2)} |z| \leq |f(z)| \leq 1 + \frac{1 - \gamma}{\sigma(\gamma, k, 2)} |z|
\end{equation}

for $\xi > 0$, $z \in U$. The result is sharp for the function

\begin{equation}
f(z) = z - \frac{1 - \gamma}{\sigma(\gamma, k, 2)} z^{2}.
\end{equation}

Letting $\xi \to 1$ in Corollary 3.11, we have

**Corollary 2.7 ([3]).** Let the function $f(z)$ defined by (1.2) be in the class $UT(\Phi, \Psi, \gamma, k)$. If $\{\sigma(\gamma, k, n)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have

\begin{equation}
1 - \frac{2(1 - \gamma)}{\sigma(\gamma, k, 2)} |z| \leq |f'(z)| \leq 1 + \frac{2(1 - \gamma)}{\sigma(\gamma, k, 2)} |z|
\end{equation}

for $0 \leq \xi < 1$, $z \in U$. The result is sharp for the function given by (3.27).

**References**


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