Angular estimates of analytic functions defined by Carlson - Shaffer linear operator

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Abstract

The object of the present paper is to derive some argument properties of analytic functions defined by the Carlson - Shaffer linear operator $L(a, c)f(z)$. Our results contain some interesting corollaries as the special cases.

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1 Introduction and definitions

Let \( A \) denote the class of functions of the form:

\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disc \( \mathcal{U} = \{ z : |z| < 1 \} \). For two functions \( f(z) \) and \( g(z) \) given by

\[
(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]

their Hadamard product (or convolution) is defined by

\[
(1.3) \quad (f \ast g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]

Define the function \( \phi(a, c; z) \) by

\[
(1.4) \quad \phi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}
\]

\( (a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}, z \in \mathcal{U}) \),

where \((\lambda)_n\) is the Pochhammer symbol given, in terms of Gamma functions,

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1)(\lambda + 2)\ldots(\lambda + n - 1), & n \in \mathbb{N} : \{1, 2, \ldots\}. \end{cases}
\]

Corresponding to the function \( \phi(a, c; z) \), Carlson and Shaffer\[1\] introduced a linear operator \( L(a, c) : A \to A \) by

\[
(1.5) \quad L(a, c)f(z) := \phi(a, c; z) \ast f(z),
\]
or, equivalently, by

\[(1.6) \quad L(a, c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n}a_{n+1}z^{n+1} \quad (z \in \mathcal{U}).\]

It follows from (1.6) that

\[(1.7) \quad z(L(a, c)f(z))' = aL(a+1, c)f(z) - (a-1)L(a, c)f(z),\]

and \(L(1, 1)f(z) = f(z),\) \(L(2, 1)f(z) =zf'(z) ,\) \(L(3, 1)f(z) =zf'(z) + \frac{1}{2}z^2f''(z).\)

Many properties of analytic functions defined by the Carlson-Shaffer linear operator were studied by (among others) Owa and Srivastava [7], Ding [3], Kim and Lee [4], Ravichandran \textit{et al.}[6] and Shanmugam \textit{et al.}[5].

In this paper we shall derive some argument properties of analytic functions defined by the linear operator \(L(a, c)f(z).\)

In order to prove our main results, we recall the following lemma:

**Lemma 1.1.** ([2]). Let \(p(z)\) be analytic in \(\mathcal{U}\) with \(p(0) = 1\) and \(p(z) \neq 0\) \((z \in \mathcal{U})\) and suppose that

\[(1.8) \quad |\arg(p(z) + \beta z p'(z))| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \alpha \beta \right) \quad (\alpha > 0, \beta > 0),\]

then we have

\[(1.9) \quad |\arg p(z)| < \frac{\pi}{2} \alpha \quad (z \in \mathcal{U}).\]
2 Main Results

Theorem 2.1. Let \( a + 1 > \mu > 0, \alpha > 0 \), \( \lambda \) is any real number, \( L(a, c) f(z)/L(a + 1, c) f(z) \neq 0 \) (\( z \in U \)) and suppose that

\[
\left| \arg \left( \frac{(a + 1)L(a, c)f(z)}{(a + 1 - \mu)L(a + 1, c)f(z)} \left[ \lambda \frac{L(a + 1, c)f(z)}{L(a, c)f(z)} - \mu \frac{L(a + 2, c)f(z)}{L(a + 1, c)f(z)} + 1 \right] \right) \right|
\]

\[
- \left( \frac{(a + 1)\lambda - a\mu}{a + 1 - \mu} \right) \left| 1 + \left( \frac{a}{p(z)} \right) - \frac{z p'(z)}{p(z)} \right| \leq \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\mu}{a + 1 - \mu} \alpha \right)
\]

(2.1)

then we have

\[
\left| \arg \left( \frac{L(a, c)f(z)}{L(a + 1, c)f(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

(2.2)

Proof. Define the function \( p(z) \) by

\[
p(z) := \frac{L(a, c)f(z)}{L(a + 1, c)f(z)}.
\]

Then \( p(z) = 1 + b_1 z + b_2 z + \cdots \) is analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) (\( z \in U \)). Also, by a simple computation, we find from (2.3) that

\[
z p'(z) = \left( \frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - \frac{z(L(a + 1, c)f(z))'}{L(a + 1, c)f(z)} \right)
\]

(2.4)

by making use of the familiar identity (1.7) in (2.4), we get

\[
\left[ \lambda \frac{L(a + 1, c)f(z)}{L(a, c)f(z)} - \mu \frac{L(a + 2, c)f(z)}{L(a + 1, c)f(z)} + 1 \right] \frac{L(a, c)f(z)}{L(a + 1, c)f(z)} = \left[ \lambda \frac{p(z)}{p(z)} - \frac{\mu}{a + 1} \left( 1 + \frac{a}{p(z)} - \frac{z p'(z)}{p(z)} \right) + 1 \right] p(z)
\]

\[
= \frac{1}{a + 1} [(a + 1)\lambda - a\mu + (a + 1 - \mu)p(z) + \mu z p'(z)]
\]
or, equivalently,

\[
\frac{(a+1)L(a,c)f(z)}{(a+1-\mu)L(a+1,c)f(z)} \left[ \lambda \frac{L(a+1,c)f(z)}{L(a,c)f(z)} - \mu \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + 1 \right]
\]

\[\tag{2.5}
- \left( \frac{(a+1)\lambda - a\mu}{a + 1 - \mu} \right) = p(z) + \frac{\mu}{a + 1 - \mu}zp'(z).
\]

The result of Theorem 2.1 now follows by an application of Lemma 1.1.

Letting \( a = c = 1 \) in Theorem 2.1, we have

**Corollary 2.2.** Let \( 2 > \mu > 0, \alpha > 0, \lambda \) is any real number, \( f(z)/zf'(z) \neq 0 \) \((z \in U)\) and suppose that

\[
\left| \arg \left( \frac{2f(z)}{(2-\mu)zf'(z)} \left[ \frac{\lambda zf'(z)}{f(z)} - \frac{\mu zf''(z)}{2f'(z)} + 1 - \mu \right] - \left( \frac{2\lambda - \mu}{2-\mu} \right) \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \frac{\tan^{-1} \left( \frac{\mu}{2-\mu} \right)}{\alpha} \right)
\]

\[\tag{2.6}
\text{then we have}
\]

\[
\left| \arg \left( \frac{f(z)}{zf'(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

**Theorem 2.3.** Let \( a \neq -1, \lambda \neq -\mu, \alpha, \lambda > 0, \delta(a+1)(\lambda + \mu) > 0, \)

\( L(a+1,c)f(z)/z \neq 0 \) \((z \in U)\) and suppose that

\[
\left| \arg \left( \frac{1}{\lambda + \mu} \left( \frac{L(a+1,c)f(z)}{z} \right) \delta \left( \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \mu \right) \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \frac{\tan^{-1} \frac{\lambda}{\delta(a+1)(\lambda+\mu)}}{\alpha} \right)
\]

\[\tag{2.8}
\text{then we have}
\]

\[
\left| \arg \left( \frac{L(a+1,c)f(z)}{z} \right) \delta \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

\[\tag{2.9}
Proof. Define the function $p(z)$ by

$$(2.10) \quad p(z) := \left( \frac{L(a + 1, c)f(z)}{z} \right) \delta.$$ 

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $U$ with $p(0) = 1$ and $p(z) \neq 0$ ($z \in U$). Also, by a simple computation and by making use of the familiar identity (1.7) we find from (2.10) that

$$(2.11) \quad \frac{L(a + 2, c)f(z)}{L(a + 1, c)f(z)} = \frac{1}{\delta(a + 1)} \frac{z p'(z)}{p(z)} + 1$$

by using (2.10) and (2.11), we get

$$\frac{1}{\lambda + \mu} \left( \frac{L(a + 1, c)f(z)}{z} \right) \delta \left( \lambda \frac{L(a + 2, c)f(z)}{L(a + 1, c)f(z)} + \mu \right)$$

$$(2.12) = p(z) + \frac{\lambda}{\delta(a + 1)(\lambda + \mu)} z p'(z)$$

Using Lemma 1.1, we obtain the required result.

Letting $a = c = 1$ in Theorem 2.3, we have

**Corollary 2.4.** Let $\lambda \neq -\mu$, $\alpha$, $\lambda > 0$, $2\delta(\lambda + \mu) > 0$, $f'(z) \neq 0$ ($z \in U$) and suppose that

$$\left| \arg \left( (f'(z)) \delta \left( 1 + \frac{\lambda z f''(z)}{2(\lambda + \mu)f'(z)} \right) \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{\alpha \tan^{-1} \frac{\lambda}{2\delta(\lambda + \mu)}}{2(\lambda + \mu)} \right)$$

$$(2.13)$$

then we have

$$(2.14) \quad |\arg (f'(z))\delta| < \frac{\pi}{2} \alpha \quad (z \in U).$$
**Theorem 2.5.** Let \( a \neq -1, \alpha, \lambda, \eta, \gamma > 0, L(a+1,c)f(z)/L(a,c)f(z) \neq 0 \ (z \in U) \) and suppose that
\[
\arg \left[ \left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^\gamma \left[ \frac{\gamma \eta}{\lambda} \left( (a + 1) \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a \frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right) + 1 \right] \right] < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda}{\eta} \right)
\]
(2.15) then we have
\[
\left| \arg \left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^\gamma \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]
(2.16)

Proof. Define the function \( p(z) \) by
\[
p(z) := \left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^\gamma.
\]
(2.17)

Then \( p(z) = 1 + b_1 z + b_2 z + \cdots \) is analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) \((z \in U)\). Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.17) that
\[
\left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^\gamma \left[ \frac{\gamma \eta}{\lambda} \left( (a + 1) \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a \frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right) + 1 \right] = p(z) + \frac{\lambda}{\eta} z p'(z)
\]
(2.18)

An application of Lemma 1.1, we obtain the required result.

Letting \( a = c = 1 \) in Theorem 2.5, we have

**Corollary 2.6.** Let \( \alpha, \lambda, \eta, \gamma > 0, zf'(z)/f(z) \neq 0 \) \((z \in U)\) and suppose that
\[
\left| \arg \left\{ \left( \frac{zf'(z)}{f(z)} \right)^\gamma \left[ \frac{\gamma \eta}{\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) + 1 \right] \right\} \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda}{\eta} \right)
\]
(2.19)
then we have

\[ |\arg\left(\frac{zf'(z)}{f(z)}\right)^\gamma| < \frac{\pi}{2}\alpha \quad (z \in \mathcal{U}). \]

Letting \( \lambda = \eta = \gamma = 1 \) in Corollary 2.6, we have

**Corollary 2.7.** Let \( 0 < \alpha \leq 1 \), \( zf'(z)/f(z) \neq 0 \) \( (z \in \mathcal{U}) \) and suppose that

\[ |\arg\left(\frac{zf'(z)}{f(z)}\right)\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right) \]

then we have

\[ |\arg\left(\frac{zf'(z)}{f(z)}\right)| < \frac{\pi}{2}\alpha \quad (z \in \mathcal{U}), \]

that is, \( f(z) \) is strongly starlike function of order \( \alpha \) in \( \mathcal{U} \).

**Theorem 2.8.** Let \( a \neq -1, \alpha, \lambda, \gamma > 0 \), \( (\gamma - \lambda)(a + 1) > 0 \), \( z/L(a + 1, c)f(z) \neq 0 \) \( (z \in \mathcal{U}) \) and suppose that

\[ |\arg\left[\frac{1}{\gamma - \lambda}\left(\frac{z}{L(a + 1, c)f(z)} - \lambda L(a + 2, c)f(z)\right)\right]| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda}{(\gamma - \lambda)(a + 1)\alpha}\right). \]

then we have

\[ |\arg\left(\frac{\lambda}{L(a + 1, c)f(z)}\right)| < \frac{\pi}{2}\alpha \quad (z \in \mathcal{U}). \]

Proof. Define the function \( p(z) \) by

\[ p(z) := \frac{z}{L(a + 1, c)f(z)} \]
Then \( p(z) = 1 + b_1z + b_2z + \cdots \) is analytic in \( U \) with \( p(0) = 1 \) and \( p(z) \neq 0 \) \((z \in U)\). Also, by a simple computation and by making use of the familiar identity \((1.7)\), we find from \((2.25)\) that

\[
(2.26) \quad \frac{1}{\gamma - \lambda} \left( \frac{z}{L(a + 1, c)f(z)} - \lambda z \frac{L(a + 2, c)f(z)}{[L(a + 1, c)f(z)]^2} \right) = p(z) + \frac{\lambda}{(\gamma - \lambda)(a + 1)}zp'(z).
\]

The result of Theorem 2.8 now follows by an application of Lemma 1.1.

Letting \( a = c = 1 \) in Theorem 2.8, we have

**Corollary 2.9.** Let \( \alpha, \lambda, \gamma > 0, (\gamma - \lambda)(a + 1) > 0, 1/f'(z) \neq 0 \) \((z \in U)\) and suppose that

\[
(2.27) \quad \left| \arg \left( \frac{1}{f'(z)} - \frac{\lambda zf''(z)}{2(\gamma - \lambda)(f'(z))^2} \right) \right| < \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda}{2(\gamma - \lambda)} \alpha \right)
\]

then we have

\[
(2.28) \quad \left| \arg \left( \frac{1}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U).
\]

**References**


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