On the Univalence of Some Integral Operators

Virgil Pescar

Abstract

In this paper some integral operators are studied and are determined conditions for the univalence of these operators.

2000 Mathematics Subject Classification: 30C45.

Key words and phrases: Univalent, Integral operator.

1 Introduction

Let $A$ be the class of the functions $f$ which are regular in the unit disc $U = \{ z \in \mathbb{C}, |z| < 1 \}$ and $f(0) = f'(0) - 1 = 0$. We denote by $S$ the class of the functions $f \in A$ which are univalent in $U$.


---

1 Received May 10, 2006
Accepted for publication (in revised form) June 12, 2006
**Theorem A.** Let \( c \) be a complex number, \( |c| \leq 1, c \neq -1 \). If \( f(z) = z + az^2 + \ldots \) is a regular function in \( U \) and
\[
|cz| + (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1}
\]
for all \( z \in U \), then the function \( f(z) \) is regular and univalent in \( U \).

Further, V. Pescar [6] gave

**Theorem B.** Let \( \alpha \) be a complex number, \( \Re \alpha > 0 \), and \( c \) a complex number, \( |c| \leq 1, c \neq -1 \) and \( f(z) = z + \cdots \) a regular function in \( U \). If
\[
|cz|^{2\alpha} + (1 - |z|^{2\alpha}) \left| \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \tag{2}
\]
for all \( z \in U \), then the function
\[
F_\alpha(z) = \left[ \alpha \int_0^z u^{\alpha-1} f'(u)du \right]^{\frac{1}{\alpha}} = z + \ldots, \tag{3}
\]
is regular and univalent in \( U \).

In this paper we will need the following theorems.

**Theorem C.** [5] Let \( f \in A \) satisfy the condition
\[
\left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < 1, \ z \in U, \tag{4}
\]
then \( f \) is univalent in \( U \).

**Theorem D.** [8] Let \( \alpha \) be a complex number, \( \Re \alpha > 0 \) and \( c \) a complex number, \( |c| \leq 1, c \neq -1 \) and \( f \in A \). If
\[
\frac{1 - |z|^{2\Re \alpha}}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c|, \tag{5}
\]
for all \( z \in U \), then for any complex number \( \beta, \ \Re \beta \geq \Re \alpha \), the function
\[
F_\beta(z) = \left[ \beta \int_0^z u^{\beta-1} f'(u)du \right]^{\frac{1}{\beta}} \tag{6}
\]
is in the class $S$.

**Schwarz Lemma.** [3] Let $f(z)$ the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M, M$ fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ z \in U_R$$

the equality (in (7) for $z \neq 0$) can hold only if $f(z) = e^{i\theta} \frac{M}{R^m}z^m$, where $\theta$ is constant.

## 2 Main results

**Theorem 1.** Let the function $g \in A$ satisfy (4), $M$ be a positiv real number fixed and $c$ be a complex number. If $\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M}\right]$, $c \neq -1$

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| (2M + 1), c \neq -1$$

and

$$|g(z)| \leq M$$

for all $z \in U$, then the function

$$G_\alpha(z) = \left[\alpha \int_0^z [g(u)]^{\alpha-1}du\right]^{\frac{1}{\alpha}}$$

is in the class $S$.

**Proof.** From (10) we have

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right)^{\alpha-1}du\right]^{\frac{1}{\alpha}}.$$
Let us consider the function

\[ f(z) = \int_0^z \left( \frac{g(u)}{u} \right)^{\alpha - 1} du. \]

The function \( f \) is regular in \( U \).

From (12) we get

\[ f'(z) = \left( \frac{g(z)}{z} \right)^{\alpha - 1}, \quad f''(z) = (\alpha - 1) \left( \frac{g(z)}{z} \right)^{\alpha - 2} \frac{zg'(z) - g(z)}{z^2} \]

and

\[
\begin{align*}
|cz^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)}| &= \\
&= |cz^{2\alpha} + (1 - |z|^{2\alpha}) \frac{\alpha - 1}{\alpha} \left( \frac{zg'(z)}{g(z)} - 1 \right)| \\
&\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left( \left| \frac{z^2 g'(z)}{g^2(z)} \right| \frac{|g(z)|}{|z|} + 1 \right)
\end{align*}
\]

for all \( z \in U \).

We have \( g(0) = 0 \) and \( |g(z)| < M \) and by the Schwarz-Lemma we obtain \( |g(z)| < M|z| \). Using (13), we have

\[
\begin{align*}
|cz^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)}| &\leq \\
&\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left[ \left( \frac{z^2 g'(z)}{g^2(z)} \right) \left| \frac{g(z)}{|z|} \right| + 1 \right] M + 1
\end{align*}
\]

From (14) and since \( g \) satisfies the condition (4) we have

\[
\begin{align*}
|cz^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)}| &\leq |c| + \left| \frac{\alpha - 1}{\alpha} \right| (2M + 1)
\end{align*}
\]

For \( \alpha \in \left[ \frac{2M+1}{2M+2}, \frac{2M+1}{2M} \right] \) we have

\[
|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (2M + 1) \leq 1
\]

and, hence, we get

\[
|cz^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)}| \leq 1, \quad z \in U.
\]
for all \( z \in U \).

From (12) we have \( f'(z) = \left( \frac{g(z)}{z} \right)^{\alpha-1} \) and by Theorem B for \( \alpha \) real number, \( \alpha > 0 \), it results that the function \( G_\alpha \) is in the class \( S \).

**Theorem 2.** Let \( g \in A \), \( \alpha \) be a real number, \( \alpha \geq 1 \), and \( c \) a complex number, \( |c| \leq \frac{1}{\alpha} \), \( c \neq -1 \). If

\[
\left| \frac{g''(z)}{g'(z)} \right| \leq 1, \quad z \in U
\]

then the function

\[
H_\alpha(z) = \alpha \int_0^z [ug'(u)]^{\alpha-1} du \right)^{\frac{1}{\alpha}}
\]

is in the class \( S \).

**Proof.** We observe that

\[
H_\alpha(z) = \left[ \alpha \int_0^z u^{\alpha-1} (g'(u))^{\alpha-1} du \right]^\frac{1}{\alpha}
\]

Let us consider the function

\[
p(z) = \int_0^z [g'(u)]^{\alpha-1} du.
\]

The function \( p \) is regular in \( U \).

From (21) we have

\[
p'(z) = (g'(z))^{\alpha-1}, \quad p''(z) = (\alpha - 1) [g'(z)]^{\alpha-2} g''(z)
\]

and we obtain

\[
\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zp''(z)}{\alpha p'(z)} \right| = \left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zg''(z)}{g'(z)} \frac{\alpha - 1}{\alpha} \right|.
\]
From (22), (18) and the conditions of theorem we get

\[(23) \quad \left| c|z|^{2\alpha} + \left( 1 - |z|^{2\alpha} \frac{zp''(z)}{\alpha p'(z)} \right) \right| \leq |c| + \frac{\alpha - 1}{\alpha} \leq 1\]

for all \( z \in U \).

By Theorem B for \( \alpha \) real number, \( \alpha \geq 1 \), and since \( p'(z) = [g'(z)]^{\alpha - 1} \) it results that the function \( H_\alpha \) is in the class \( S \).

**Theorem 3.** Let \( g \in A \) satisfies (4), \( \alpha \) be a complex number, \( M > 1 \) fixed, \( \text{Re}\alpha > 0 \) and \( c \) be a complex number, \( |c| < 1 \). If

\[(24) \quad |g(z)| \leq M\]

for all \( z \in U \), then for any complex number \( \beta \)

\[(25) \quad \text{Re}\beta \geq \text{Re}\alpha \geq \frac{2M + 1}{|\alpha| (1 - |c|)}\]

the function

\[(26) \quad H_\beta(z) = \left[ \beta \int_0^z u^{\beta - 1} \left( \frac{g(u)}{u} \right)^{\frac{1}{\alpha}} \, du \right]^{\frac{1}{\beta}}.\]

is in the class \( S \).

**Proof.** Let us consider the function

\[(27) \quad f(z) = \int_0^z \left( \frac{g(u)}{u} \right)^{\frac{1}{\alpha}} \, du.\]

The function \( f \) is regular in \( U \). From (27) we have:

\[
\begin{align*}
f'(z) &= \left( \frac{g(z)}{z} \right)^{\frac{1}{\alpha}}, \\
f''(z) &= \frac{1}{\alpha} \left( \frac{g(z)}{z} \right)^{\frac{1}{\alpha} - 1} \frac{zg'(z) - g(z)}{z^2}
\end{align*}
\]
and
\[
(28) \quad \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2Re\alpha}}{|\alpha|Re\alpha} \left| \frac{zg'(z)}{g(z)} \right| + \frac{1 - |z|^{2Re\alpha}}{|\alpha|Re\alpha}
\]
for all \( z \in U \) and hence, we have
\[
(29) \quad \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2Re\alpha}}{|\alpha|Re\alpha} \left( \left| \frac{z^2g'(z)}{g^2(z)} \right| \cdot \left| \frac{g(z)}{z} \right| + 1 \right)
\]
for all \( z \in U \).

By the Schwarz-Lemma also \( |g(z)| \leq M|z|, \ z \in U \) and using (29) we obtain
\[
(30) \quad \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2Re\alpha}}{|\alpha|Re\alpha} \left( \left| \frac{z^2g'(z)}{g^2(z)} \right| - 1 + 1 \right) M + 1
\]
for all \( z \in U \).

From (30) and since \( g \) satisfies the condition (4) we get
\[
(31) \quad \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2Re\alpha}}{|\alpha|Re\alpha} \frac{2M + 1}{|\alpha|} \leq \frac{2M + 1}{|\alpha|}
\]
for all \( z \in U \).

From (25) we have \( \frac{2M + 1}{|\alpha|Re\alpha} \leq 1 - |c| \) and using (31) we obtain
\[
(32) \quad \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c|.
\]

From (27) we obtain \( f'(z) = \left( \frac{g(z)}{z} \right)^{\frac{1}{\alpha}} \) and using (32) by Theorem D we conclude that the function \( H_\beta \) is in the class \( S \).

References


"Transilvania” University of Brașov
Faculty of Mathematics and Computer Science
Department of Mathematics
Str. Iuliu Maniu Nr. 50, Brașov - 500091, ROMANIA
E-mail address: virgilsescar@unitbv.ro