Subclasses of starlike functions associated with some hyperbola\textsuperscript{1}

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Abstract

In this paper we define some subclasses of starlike functions associated with some hyperbola by using a generalized Sălăgean operator and we give some properties regarding these classes.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$, $A = \{ f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0 \}$, $\mathcal{H}_u(U) = \{ f \in \mathcal{H}(U) : f \text{ is univalent in } U \}$ and $S = \{ f \in A : f \text{ is univalent in } U \}$.

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Let $D^n$ be the Sălăgean differential operator (see [12]) defined as:

$$D^n : A \to A, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = D f(z) = zf'(z), \quad D^n f(z) = D(D^{n-1} f(z)).$$

**Remark 1.1.** If $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$ then

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j.$$

We recall here the definition of the well-known class of starlike functions

$$S^* = \left\{ f \in A : \Re \frac{zf'(z)}{f(z)} > 0, \quad z \in U \right\}.$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$f(z) = L_a F(z) = \frac{1 + a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C}, \quad \Re a \geq 0. \tag{1}$$

Generalizations of the Libera-Pascu integral operator was studied by many mathematicians such as P.T. Mocanu in [7], E. Drăghici in [6] and D. Breaz in [5].

**Definition 1.1.**[4] Let $n \in \mathbb{N}$ and $\lambda \geq 0$. We denote with $D^n_\lambda$ the operator defined by

$$D^n_\lambda : A \to A,$$

$$D^n_\lambda f(z) = D f(z), \quad D^1_\lambda f(z) = (1 - \lambda)f(z) + \lambda zf'(z) = D_\lambda f(z),$$

$$D^n_\lambda f(z) = D_\lambda \left( D^n_{\lambda^{-1}} f(z) \right).$$

**Remark 1.2.**[4] We observe that $D^n_\lambda$ is a linear operator and for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$D^n_\lambda f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^n a_j z^j.$$
Also, it is easy to observe that if we consider $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator.

The next theorem is result of the so-called ”admissible functions method” introduced by P.T. Mocanu and S.S. Miller (see [8], [9], [10]).

**Theorem 1.1.** Let $h$ convex in $U$ and $\text{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and $p$ satisfied the Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z), \quad \text{then} \quad p(z) < h(z).$$

In [1] is introduced the following operator:

**Definition 1.2.** Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by $D_{\lambda}^{\beta}$ the linear operator defined by

$$D_{\lambda}^{\beta} : A \to A,$$

$$D_{\lambda}^{\beta} f(z) = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta} a_j z^j.$$

**Remark 1.3.** It is easy to observe that for $\beta = n \in \mathbb{N}$ we obtain the Al-Oboudi operator $D_{\lambda}^{n}$ and for $\beta = n \in \mathbb{N}$, $\lambda = 1$ we obtain the Sălăgean operator $D^{n}$.

The purpose of this note is to define some subclasses of starlike functions associated with some hyperbola by using the operator $D_{\lambda}^{\beta}$ defined above and to obtain some properties regarding these classes.
2 Preliminary results

Definition 2.1. [13] A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies

$$\left| \frac{zf'(z)}{f(z)} - 2\alpha \left( \sqrt{2} - 1 \right) \right| < \text{Re} \left\{ \sqrt{2} \frac{zf'(z)}{f(z)} \right\} + 2\alpha \left( \sqrt{2} - 1 \right),$$

for some $\alpha$ ($\alpha > 0$) and for all $z \in U$.

Remark 2.1. Geometric interpretation:

Let $\Omega(\alpha) = \left\{ \frac{zf'(z)}{f(z)} : z \in U, f \in SH(\alpha) \right\}$.

Then $\Omega(\alpha) = \left\{ w = u + i \cdot v : v^2 < 4\alpha u + u^2, u > 0 \right\}$. Note that $\Omega(\alpha)$ is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin.

Definition 2.2. [3] Let $f \in S$ and $\alpha > 0$. We say that the function $f$ is in the class $SH_n(\alpha)$, $n \in \mathbb{N}$, if

$$\left| \frac{D^{n+1}f(z)}{D^nf(z)} - 2\alpha \left( \sqrt{2} - 1 \right) \right| < \text{Re} \left\{ \sqrt{2} \frac{D^{n+1}f(z)}{D^nf(z)} \right\} + 2\alpha \left( \sqrt{2} - 1 \right), z \in U.$$

Remark 2.2. Geometric interpretation: If we denote with $p_\alpha$ the analytic and univalent functions with the properties $p_\alpha(0) = 1$, $p_\alpha'(0) > 0$ and $p_\alpha(U) = \Omega(\alpha)$ (see Remark 2.1), then $f \in SH_n(\alpha)$ if and only if $\frac{D^{n+1}f(z)}{D^nf(z)} < p_\alpha(z)$, where the symbol $<$ denotes the subordination in $U$.

We have $p_\alpha(z) = (1 + 2\alpha) \sqrt{\frac{1 + bz}{1 - z}} - 2\alpha$, $b = b(\alpha) = \frac{1 + 4\alpha - 4\alpha^2}{(1 + 2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $\text{Im} \sqrt{w} \geq 0$.

Theorem 2.1. [3] If $F(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$, and $f(z) = L_\alpha F(z)$, where $L_\alpha$ is the integral operator defined by (1), then $f(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$.
Theorem 2.2. [3] Let $n \in \mathbb{N}$ and $\alpha > 0$. If $f \in SH_{n+1}(\alpha)$ then $f \in SH_n(\alpha)$.

3 Main results

Definition 3.1. Let $\beta \geq 0$, $\lambda \geq 0$, $\alpha > 0$ and $p_\alpha(z) = (1 + 2\alpha)\sqrt{1 + \frac{bz}{1 - z}} - 2\alpha$, where $b = b(\alpha) = \frac{1 + 4\alpha - 4\alpha^2}{(1 + 2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $\text{Im}\sqrt{w} \geq 0$. We say that a function $f(z) \in S$ is in the class $SH_{\beta,\lambda}(\alpha)$ if

$$\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)} \prec p_\alpha(z), \quad z \in U.$$ 

Remark 3.1. Geometric interpretation: $f(z) \in SH_{\beta,\lambda}(\alpha)$ if and only if $\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)}$ take all values in the domain $\Omega(\alpha)$ which is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin (see Remark 2.1 and Remark 2.2).

Remark 3.2. It is easy to observe that for $\beta = n \in \mathbb{N}$ and $\lambda = 1$ we obtain in the above definition we obtain the class $SH_n(\alpha)$ studied in [3] and for $\lambda = 1$, $\beta = 0$ we obtain the class $SH(\alpha)$ studied in [13].

Theorem 3.1. Let $\beta \geq 0$, $\alpha > 0$ and $\lambda > 0$. We have

$$SH_{\beta+1,\lambda}(\alpha) \subset SH_{\beta,\lambda}(\alpha).$$

Proof. Let $f(z) \in SH_{\beta+1,\lambda}(\alpha)$.

With notation

$$p(z) = \frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)}, \quad p(0) = 1,$$
we obtain

\begin{equation}
\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^\beta f(z)}, \quad \frac{D_\lambda^\beta f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^\beta f(z)}
\end{equation}

Also, we have

\[
\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^\beta f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta+2} a_j z^j}{z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta} a_j z^j}
\]

and

\[
z p'(z) = \frac{z \left( \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \right)' - \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} \cdot \frac{z \left( \frac{D_\lambda^\beta f(z)}{D_\lambda^\beta f(z)} \right)'}{D_\lambda^\beta f(z)}}{D_\lambda^\beta f(z)}
\]

\[
- p(z) \cdot \frac{z \left( 1 + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda)^{\beta+1} j a_j z^{j-1} \right)}{D_\lambda^\beta f(z)}
\]

or

\begin{equation}
z p'(z) = \frac{z + \sum_{j=2}^{\infty} j \left( 1 + (j - 1)\lambda \right)^{\beta+1} a_j z^j}{D_\lambda^\beta f(z)}
\end{equation}

We have

\[
z + \sum_{j=2}^{\infty} j \left( 1 + (j - 1)\lambda \right)^{\beta+1} a_j z^j =
\]
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\[ z + \sum_{j=2}^{\infty} ((j - 1) + 1) (1 + (j - 1)\lambda) a_j z^j = \]

\[ = z + \sum_{j=2}^{\infty} (1 + (j - 1)\lambda) a_j z^j + \sum_{j=2}^{\infty} (j - 1) (1 + (j - 1)\lambda) a_j z^j = \]

\[ = z + D_{\lambda}^{\beta+1} f(z) - z + \sum_{j=2}^{\infty} (j - 1) (1 + (j - 1)\lambda) a_j z^j = \]

\[ = D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda} \sum_{j=2}^{\infty} (1 + (j - 1)\lambda - 1) (1 + (j - 1)\lambda) a_j z^j = \]

\[ = D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} \sum_{j=2}^{\infty} (1 + (j - 1)\lambda) a_j z^j + \frac{1}{\lambda} \sum_{j=2}^{\infty} (1 + (j - 1)\lambda) a_j z^j = \]

\[ = D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} D_{\lambda}^{\beta+1} f(z) - \frac{1}{\lambda} D_{\lambda}^{\beta+2} f(z) = \]

\[ = \frac{\lambda - 1}{\lambda} D_{\lambda}^{\beta+1} f(z) + \frac{1}{\lambda} D_{\lambda}^{\beta+2} f(z) = \]

\[ = \frac{1}{\lambda} \left( (\lambda - 1) D_{\lambda}^{\beta+1} f(z) + D_{\lambda}^{\beta+2} f(z) \right). \]

Similarly we have

\[ z + \sum_{j=2}^{\infty} j (1 + (j - 1)\lambda) a_j z^j = \frac{1}{\lambda} \left( (\lambda - 1) D_{\lambda}^{\beta} f(z) + D_{\lambda}^{\beta+1} f(z) \right). \]

From (3) we obtain

\[ z p'(z) = \]

\[ = \frac{1}{\lambda} \left( \frac{(\lambda - 1) D_{\lambda}^{\beta+1} f(z) + D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta} f(z)} - p(z) \frac{(\lambda - 1) D_{\lambda}^{\beta} f(z) + D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} \right). \]
\[
= \frac{1}{\lambda} \left( (\lambda - 1)p(z) + \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta} f(z)} - p(z) ((\lambda - 1) + p(z)) \right) = \\
= \frac{1}{\lambda} \left( \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta} f(z)} - p(z)^2 \right)
\]

Thus
\[
\lambda z p'(z) = \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta} f(z)} - p(z)^2
\]
or
\[
\frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta} f(z)} = p(z)^2 + \lambda z p'(z).
\]

From (2) we obtain
\[
\frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} = \frac{1}{p(z)} \left( p(z)^2 + \lambda z p'(z) \right) = p(z) + \lambda \frac{z p'(z)}{p(z)},
\]
where \(\lambda > 0\).

From \(f(z) \in SH_{\beta+1,\lambda}(\alpha)\) we have
\[
p(z) + \lambda \frac{z p'(z)}{p(z)} < p_{\alpha}(z),
\]
with \(p(0) = p_{\alpha}(0) = 1\), \(\alpha > 0\), \(\beta \geq 0\), \(\lambda > 0\), and \(\text{Re } p_{\alpha}(z) > 0\) from here construction. In this conditions from Theorem 1.1, we obtain
\[
p(z) < p_{\alpha}(z)
\]
or
\[
\frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} < p_{\alpha}(z).
\]
This means \(f(z) \in SH_{\beta,\lambda}(\alpha)\).

**Theorem 3.2.** Let \(\beta \geq 0\), \(\alpha > 0\) and \(\lambda \geq 1\). If \(F(z) \in SH_{\beta,\lambda}(\alpha)\) then \(f(z) = L_{\alpha} F(z) \in SH_{\beta,\lambda}(\alpha)\), where \(L_{\alpha}\) is the Libera-Pascu integral operator defined by (1).
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**Proof.** From (1) we have

\[(1 + a)F(z) = af(z) + zf'(z)\]

and, by using the linear operator \(D^{\beta+1}_\lambda\), we obtain

\[(1 + a)D^{\beta+1}_\lambda F(z) = aD^{\beta+1}_\lambda f(z) + D^{\beta+1}_\lambda \left(z + \sum_{j=2}^{\infty} ja_j z^j\right) = aD^{\beta+1}_\lambda f(z) + z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} ja_j z^j\]

We have (see the proof of the above theorem)

\[z + \sum_{j=2}^{\infty} j (1 + (j-1)\lambda)^{\beta+1} a_j z^j = \frac{1}{\lambda} \left( (\lambda - 1)D^{\beta+1}_\lambda f(z) + D^{\beta+2}_\lambda f(z) \right)\]

Thus

\[(1 + a)D^{\beta+1}_\lambda F(z) = aD^{\beta+1}_\lambda f(z) + \frac{1}{\lambda} \left( (\lambda - 1)D^{\beta+1}_\lambda f(z) + D^{\beta+2}_\lambda f(z) \right) = \left( a + \frac{\lambda - 1}{\lambda} \right) D^{\beta+1}_\lambda f(z) + \frac{1}{\lambda} D^{\beta+2}_\lambda f(z)\]

or

\[\lambda(1 + a)D^{\beta+1}_\lambda F(z) = ((a + 1)\lambda - 1) D^{\beta+1}_\lambda f(z) + D^{\beta+2}_\lambda f(z).\]

Similarly, we obtain

\[\lambda(1 + a)D^{\beta}_\lambda F(z) = ((a + 1)\lambda - 1) D^{\beta}_\lambda f(z) + D^{\beta+1}_\lambda f(z).\]

Then

\[\frac{D^{\beta+1}_\lambda F(z)}{D^{\beta}_\lambda F(z)} = \frac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} \cdot \frac{D^{\beta+1}_\lambda f(z)}{D^{\beta}_\lambda f(z)} + ((a + 1)\lambda - 1) \cdot \frac{D^{\beta+1}_\lambda f(z)}{D^{\beta}_\lambda f(z)}.\]
With notation
\[ \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} = p(z), \quad p(0) = 1, \]
we obtain
\[ D_{\lambda}^{\beta+1} F(z) = \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot p(z) + ((a + 1)\lambda - 1) \cdot p(z). \]

We have (see the proof of the above theorem)
\[ \lambda z p'(z) = \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} - p(z)^2 = \]
\[ = \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} \cdot p(z) - p(z)^2. \]

Thus
\[ \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} = \frac{1}{p(z)} \cdot (p(z)^2 + \lambda z p'(z)). \]

Then, from (4), we obtain
\[ \frac{D_{\lambda}^{\beta+1} F(z)}{D_{\lambda}^{\beta} F(z)} = \frac{p(z)^2 + \lambda z p'(z) + ((a + 1)\lambda - 1) p(z)}{p(z) + ((a + 1)\lambda - 1)} = \]
\[ = p(z) + \frac{z p'(z)}{p(z) + ((a + 1)\lambda - 1)}, \]
where \( a \in \mathbb{C}, \ Re a \geq 0, \ \beta \geq 0, \text{ and } \lambda \geq 1. \)

From \( F(z) \in SH_{\beta,\lambda}(\alpha) \) we have
\[ p(z) + \frac{z p'(z)}{\frac{1}{\lambda} (p(z) + ((a + 1)\lambda - 1))} \prec p_\alpha(z), \]
where \( a \in \mathbb{C}, \ Re a \geq 0, \ \alpha > 0, \ \beta \geq 0, \ \lambda \geq 1, \text{ and from her construction, we have } Re p_\alpha(z) > 0. \text{ In this conditions we have from Theorem 1.1 we obtain} \]
\[ p(z) \prec p_\alpha(z) \]
or

\[ \frac{D^{\beta+1}_{\lambda}f(z)}{D^\beta_{\lambda}f(z)} \prec p_\alpha(z). \]

This means \( f(z) = L_\alpha F(z) \in SH_{\beta,\lambda}(\alpha). \)

**Remark 3.3.** If we consider \( \beta = n \in \mathbb{N} \) in the previously results we obtain the Theorem 3.1 and Theorem 3.2 from [2].

**References**


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