On Univalence Criteria

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Abstract

By means of a new univalence criterion for the analytic functions in the open unit disk $U$ based upon the Becker's criterion, but which doesn't contain $|z|$, we give another criterion similar with the one given by Avhadiev F.G. and Aksentiev L.A.

Also using the above mentioned criterion, some univalence of the integral operators are prooved.

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1 Introduction

Let $A$ the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. Let $S$ denote the subclass of $A$ consisting of all functions $f(z)$ which are univalent in $U$. For $f \in A$ and $g \in A$, we say that the function $f(z)$ is subordinate to $g(z)$, written by $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$.
We need the following theorems due by Avhadiev F.G. and Aksentiev L.A. respectively, N.N.Pascu and V.Pescar.

**Theorem A.** [1] Let $f, g \in A$. If

\begin{equation}
(1 - |z|^2) \left| \frac{zg''(z)}{g'(z)} \right| \leq 1, \quad z \in U
\end{equation}

and $\log f'(z) < \log g'(z)$, $\log f'(0) = \log g'(0) = 0$ then the function $f$ is in $S$.

**Theorem B.** [4] Let $\alpha \in \mathbb{C}$, $\text{Re} (\alpha) \geq 0$. If $f \in A$ and

\begin{equation}
\frac{1 - |z|^{2\text{Re} (\alpha)}}{\text{Re} (\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\end{equation}

then the function

\begin{equation}
F_{\alpha}(z) = \left[ \alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{1/\alpha}
\end{equation}

belong to the class $S$.

**Theorem C.** [5] Let $\alpha, \beta, \gamma$ be complex numbers and $h \in S$. If

$$\text{Re}(\beta) \geq \text{Re} (\alpha) > 0$$

and

$$|\gamma| \leq \frac{\text{Re} (\alpha)}{2} \quad \text{for} \quad \text{Re} (\alpha) \in \left(0, \frac{1}{2}\right)$$

$$|\gamma| \leq \frac{1}{4} \quad \text{for} \quad \text{Re} (\alpha) \in \left[\frac{1}{2}, +\infty\right]$$

then the function

\begin{equation}
G_{\beta,\gamma}(z) = \left[ \beta \int_0^z u^{\beta-1} \left( \frac{h(u)}{u} \right)^\gamma du \right]^{1/\beta}
\end{equation}

belong also to the class $S$.

**Lemma D.** (Caratheodory) Let $g \in A$, and let, $M > 0$. 

If \( \text{Re} \ (g(z)) \leq M \), for any \( z \in U \) then

\[
(1 - |z|)|g(z)| \leq 2M|z| \quad z \in U
\]

**Proof.** Let us define the function \( h(z) \) by

\[
h(z) = \frac{g(z)}{2M - g(z)}
\]

Then \( h(z) \in A \) and \( |h(z)| \leq 1 \), \( z \in U \) because

\[
|g(z)| \leq |2M - g(z)|
\]

According to the Schwarz’s Lemma we have

\[
|h(z)| \leq |z| \quad (\forall \ z \in U)
\]

that is

\[
|g(z)| \leq |z| \cdot |2M - g(z)| \leq |z|(2M + |g(z)|)
\]

This implies that

\[
(1 - |z|)|g(z)| \leq 2M|z|
\]

## 2 Main Results

First we give a univalence criterion based on the Becker’s criterion but which doesn’t use the modulus of \( z \). For this reason it is easily used for practical applications. In the second part, the Lemma D and this criterion are used to obtain several univalence criteria analogous to those given by Avhadiev and Aksentiev [1], Pascu and Pescar [5].

**Theorem 1.** [3] If \( f \in A \) satisfies for some \( \theta \in [0, 2\pi] \) the inequality

\[
\text{Re} \left[ e^{i\theta} \frac{z f''(z)}{f'(z)} \right] \leq \frac{1}{4},
\]
then $f \in S$

**Proof.** If we take

$$g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$$

in Lemma D, then we have

$$(1 - |z|) \left| \frac{zf''(z)}{f'(z)} \right| \leq 2 \cdot \frac{1}{4} |z| = \frac{|z|}{2}$$

In addition, we see that

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| = (1 + |z|)(1 - |z|) \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 + |z|) \frac{|z|}{2} \leq 1$$

According to Becker’s univalence criterion [2] we conclude that $f \in S$.

**Theorem 2.** Let $f, g \in A$. If for some $\theta \in [0, 2\pi]$ the inequality

$$\text{Re} \left[ e^{i\theta} \frac{zf''(z)}{g'(z)} \right] \leq \frac{1}{4} \quad z \in U$$

is valid and $\log f'(z) < \log g'(z)$, $\log f'(0) = \log g'(0) = 0$ then $f$ is in $S$, $\forall \theta \in [0, 2\pi]$.

**Proof.** If we take $g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$ in Lemma D and using a similar way as in Theorem 1 we obtain the condition (1). According to Theorem A, the conclusion of Theorem 2 follows immediately.

**Theorem 3.** Let $f \in A$, $\alpha \in C$, $\text{Re} (\alpha) > 0$. If for some $\theta \in [0, 2\pi]$ the inequality

$$(6) \quad \text{Re} \left[ e^{i\theta} \frac{zf''(z)}{f'(z)} \right] \leq \begin{cases} \text{Re} \left( \frac{\alpha}{2} \right) & \text{for } 0 < \text{Re} (\alpha) < 1 \\ \frac{1}{4} & \text{for } \text{Re} (\alpha) \geq 1 \end{cases} \quad z \in U$$

is valid, then the function

$$F_\alpha(z) = \left[ \alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{1/\alpha}$$
is in $S$, for all $\theta \in [0, 2\pi]$.

**Proof.** We consider two cases:

a) $\text{Re} (\alpha) \geq 1$.

It is easy to observe that the function $h : (0, \infty) \to \mathbb{R}$

$$h(x) = \frac{1 - a^{2x}}{x} \quad (0 < a < 1)$$

is a decreasing function, and that, if we take $z \in U$, $a = |z|$ then

$$\frac{1 - |z|^{2\text{Re} (\alpha)}}{\text{Re} (\alpha)} \leq 1 - |z|^2 \quad (7)$$

If we put $g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$, and $M = \frac{1}{4}$ in Lemma D, then we obtain the inequality (5). According to (7) we have

$$\frac{1 - |z|^{2\text{Re} (\alpha)}}{\text{Re} (\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (8)$$

b) $0 < \text{Re} (\alpha) < 1$. The function $q(x) = 1 - a^{2x}$, $0 < a < 1$ is a increasing function, and for $a = |z|$, $z \in U$ one obtains

$$1 - |z|^{2\text{Re} (\alpha)} \leq 1 - |z|^2 \quad (0 < \text{Re} (\alpha) \leq 1) \quad (9)$$

Now if we take $M = \frac{\text{Re} (\alpha)}{4}$ in Lemma D, then

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq \text{Re} (\alpha)$$

According to (9) we have

$$(1 - |z|^{2\text{Re} (\alpha)}) \left| \frac{zf''(z)}{f'(z)} \right| \leq (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq \text{Re} (\alpha)$$

In the conclusion for all $\alpha \in C$ with $\text{Re} (\alpha) > 0$ the condition (6) implies the inequality (2) from Theorem B, that is the function $F_\alpha$ from (3) it is univalent. This completes the proof of Theorem 3.
**Theorem 4.** Let be $\alpha, \beta, \gamma$ complex numbers so that

$$Re \ (\beta) \geq Re \ (\alpha) > 0$$

and

$$|\gamma| \leq \frac{Re \ (\alpha)}{2} \quad \text{for} \ Re \ (\alpha) \in \left(0, \frac{1}{2}\right)$$

$$|\gamma| \leq \frac{1}{4} \quad \text{for} \ Re \ (\alpha) \in \left[\frac{1}{2}, +\infty\right)$$

If $h \in A$ and for some $\theta \in [0, 2\pi]$,

$$Re \left[ e^{i\theta} \frac{zh''(z)}{h'(z)} \right] \leq \frac{1}{4} \quad (z \in U),$$

then the function

$$(10) \quad G_{\beta,\alpha}(z) = \left[ \beta \int_{0}^{z} u^{\beta-1} \left( \frac{h(u)}{u} \right)^{\gamma} \, du \right]^{1/\beta}$$

belong to the class $S$.

**Proof.** For the function $\left( \frac{h(z)}{z} \right)^{\gamma}$ in (10), we can choose the regular branch which is equal to 1 at the origin. According to the Theorem 1 and Theorem C imply the conclusion of the Theorem 4.

**Remark.** In all above univalence criteria the hypothesis have conditions which do not contain $|z|$ that is, these are more practical than other similar criteria.

**References**


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