Integral Representations and its Applications in Clifford Analysis

Zhang Zhongxiang

Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In this paper, we mainly study the integral representations for functions $f$ with values in a universal Clifford algebra $C(V_{n,n})$, where $f \in \Lambda(f, \Omega)$,

$$\Lambda(f, \Omega) = \left\{ f \mid f \in C^\infty(\Omega, C(V_{n,n})), \max_{x \in \Omega} |D^j f(x)| = O(M^j) (j \to +\infty), \right\}.$$ 

The integral representations of $T_i f$ are also given. Some properties of $T_i f$ and $\Pi f$ are shown. As applications of the higher order Pompeiu formula, we get the solutions of the Dirichlet problem and the inhomogeneous equations $D^k u = f$.

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1 Introduction and Preliminaries

Integral representation formulas of Cauchy-Pompeiu type expressing complex valued, quaternionic and Clifford algebra valued functions have been well developed in [1-9, 12-19, 21, 24, 25 etc.]. These integral representation formulas serve to solve boundary value problems for partial differential equations. In [2, 3], H. Begehr gave the different integral representation formulas for functions with values in a Clifford algebra \( C(V_{n,0}) \), the integral operators provide particular weak solutions to the inhomogeneous equations \( \partial^k \omega = f, \Delta^k \omega = g \) and \( \partial \Delta^k \omega = h \). In [5, 24], the higher order Cauchy-Pompeiu formulas for functions with values in a universal Clifford algebra \( C(V_{n,n}) \) are obtained. In [16], G.N. Hile gave the detailed properties of the \( T \)-operator by following the techniques of Vekua. In [14, 15], K. Gürlebeck gave many properties of the \( \Pi \)-operator. In [18], H. Malonek and B. Müller gave some properties of the vectorial integral operator \( \vec{\Pi} \). In [7, 19, 21], the integral representations related with the Helmholtz operator are given, the weak solutions of the inhomogeneous equations \( L^k u = f \) and \( L^k u = f \), \( k \geq 1 \), are obtained, where \( Lu = Du + uh \) and \( L^k u = uD - hu \), \( h = \sum_{i=1}^{n} h_ie_i \), \( D \) is the Dirac operator. In this paper, we shall continue to study the properties of Cauchy-Pompeiu operator, higher order Cauchy-Pompeiu operator and \( \Pi \) operator for \( f \in \Lambda(f, \Omega) \), where

\[
\Lambda(f, \Omega) = \left\{ f \mid f \in C^\infty(\Omega, C(V_{n,n})), \max_{x \in \Omega} |D^j f(x)| = O(M^j) (j \to +\infty), \text{for some } M, 0 < M < +\infty \right\},
\]

the integral representations of \( T_i f \) are given, some properties of \( T_i f \) and \( \Pi f \) are shown. As applications, we get the solutions of the Dirichlet problem.
and the inhomogeneous equations $D^k u = f$ which are not in weak sense as in [2, 25].

Let $V_{n,s}(0 \leq s \leq n)$ be an $n$–dimensional ($n \geq 1$) real linear space with basis \{\(e_1, e_2, \cdots, e_n\), $C(V_{n,s})$ be the $2^n$–dimensional real linear space with basis
\[
\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\},
\]
where $N$ stands for the set \{1, $\cdots$, $n$\} and $\mathcal{P}N$ denotes the family of all order-preserving subsets of $N$ in the above way. We denote $e_\emptyset$ as $e_0$ and $e_A$ as $e_{h_1\cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. The product on $C(V_{n,s})$ is defined by
\[
(1) \quad \begin{cases}
    e_A e_B = (-1)^{#((A \cap B) \setminus S)} (-1)^{P(A,B)} e_{A \triangle B}, & \text{if } A, B \in \mathcal{P}N; \\
    \lambda \mu = \sum_{A \in \mathcal{P}N} \sum_{B \in \mathcal{P}N} \lambda_A \mu_B e_A e_B, & \text{if } \lambda = \sum_{A \in \mathcal{P}N} \lambda_A e_A, \mu = \sum_{B \in \mathcal{P}N} \mu_B e_B.
\end{cases}
\]
where $S$ stands for the set \{1, $\cdots$, $s$\}, $#(A)$ is the cardinal number of the set $A$, the number $P(A, B) = \sum_{j \in B} P(A, j)$, $P(A, j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A \triangle B$ is also order-preserving in the above way, and $\lambda_A \in \mathcal{R}$ is the coefficient of the $e_A$–component of the Clifford number $\lambda$. We also denote $\lambda_A$ as $[\lambda]_A$, for abbreviaty, we denote $\lambda_{\{i\}}$ as $[\lambda]_i$. It follows at once from the multiplication rule (1) that $e_0$ is the identity element written now as 1 and in particular,
\[
(2) \quad \begin{cases}
    e_i^2 = 1, & \text{if } i = 1, \cdots, s, \\
    e_j^2 = -1, & \text{if } j = s + 1, \cdots, n, \\
    e_i e_j = -e_j e_i, & \text{if } 1 \leq i < j \leq n, \\
    e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } 1 \leq h_1 < h_2 \cdots, < h_r \leq n.
\end{cases}
\]
Thus $C(V_{n,s})$ is a real linear, associative, but non-commutative algebra and it is called the universal Clifford algebra over $V_{n,s}$.

Frequent use will be made of the notation $\mathcal{R}^n_z$ where $z \in \mathcal{R}^n$, which means to remove $z$ from $\mathcal{R}^n$. In particular $\mathcal{R}^n_0 = \mathcal{R}^n \setminus \{0\}$.

Let $\Omega$ be an open non empty subset of $\mathcal{R}^n$, since we shall only consider the case of $s = n$ in this paper, we shall only consider the operator $D$ which is written as

$$D = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{n,n})) \to C^{(r-1)}(\Omega, C(V_{n,n})).$$

Let $f$ be a function with value in $C(V_{n,n})$ defined in $\Omega$, the operator $D$ acts on the function $f$ from the left and from the right being governed by the rule

$$D[f] = \sum_{k=1}^{n} \sum_A e_k e_A \frac{\partial f_A}{\partial x_k}, \quad [f]D = \sum_{k=1}^{n} \sum_A e_A e_k \frac{\partial f_A}{\partial x_k},$$

An involution is defined by

$$e_A = (-1)^{\sigma(A) + \#(A \cap S)} e_A, \quad \lambda \mu = \mu \lambda,$$

where $\sigma(A) = \#(A)(\#(A) + 1)/2$. From (1) and (3), we have

$$\begin{cases} 
\overline{e_i} = e_i, & \text{if } i = 0, 1, \cdots, s, \\
\overline{e_j} = -e_j, & \text{if } j = s + 1, \cdots, n, \\
\overline{\lambda \mu} = \overline{\mu} \overline{\lambda}, & \text{for any } \lambda, \mu \in C(V_{n,s}).
\end{cases}$$

The $C(V_{n,n})$–valued $(n-1)$–differential form

$$d\sigma = \sum_{k=1}^{n} (-1)^{k-1} e_k d\overline{x}_k^N$$
is exact, where
\[ d\tilde{x}_k^N = dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^n. \]

## 2 Integral Representations

In this section, we shall give the integral representations for \( f \) and \( T_i f \), \( i \geq 1 \), \( f \in \Lambda(f, \Omega) \), where

\[ \Lambda(f, \Omega) = \left\{ f \in C^\infty(\overline{\Omega}, C(V_{n,n})), \max_{x \in \Omega} |D^j f(x)| = O(M^j) (j \to +\infty), \text{for some } M, 0 < M < +\infty \right\}. \]

In [5], [24] the kernel functions

(5)

\[ H_j^*(x) = \begin{cases} \frac{A_j}{\omega_n \rho^n(x)} \frac{x^j}{\rho(x)}, & n \text{ is odd;} \\ \frac{A_j}{\omega_n \rho^n(x)} \frac{x^j}{\rho(x)}, & 1 \leq j < n, \ n \text{ is even;} \\ \frac{A_{j-1}}{2\omega_n} \log(x^2), & j = n, \ n \text{ is even;} \\ \frac{A_{n-1}}{2\omega_n} C_{l,0} \frac{e^{l}}{\log(x^2)} \left( \log(x^2) - 2 \sum_{i=0}^{l-1} \frac{C_{i+1,0}}{C_{i,0}} \right), & j = n + l, l > 0, \ n \text{ is even;} \end{cases} \]

are constructed for any \( j \geq 1 \), where \( x = \sum_{k=1}^{n} x_k e_k \), \( \rho(x) = \left( \sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}} \), \( \omega_n \) denotes the area of the unit sphere in \( \mathbb{R}^n \), and

(6) \[ A_j = \frac{1}{2^{[\frac{j+1}{2}] \frac{[\frac{j-1}{2}]}{2^{n}}} \prod_{r=1}^{\frac{j}{2}} (2r - n)}, 1 \leq j < n (n \text{ is even}), j \in N^*(n \text{ is odd}), \]
Lemma 1. (Higher order Cauchy-Pompeiu formula) (see [24]) Suppose that $M$ is an $n$–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega \subset \mathbb{R}^n$, $f \in C^{(r)}(\Omega, C(V_{n,n}))$, $r \geq k$, moreover $\partial M$ is given the induced orientation, for each $j = 1, \cdots, k$, $H_j^*(x)$ is as above. Then, for $z \in M$

\begin{equation}
\sum_{j=0}^{k-1} (-1)^j \int_{\partial M} H_{j+1}^*(x-z) d\sigma_x D^j f(x) + (-1)^k \int_M H_k^*(x-z) D^k f(x) dx^N.
\end{equation}

In the following, $\Omega$ is supposed to be an open non empty subset of $\mathbb{R}^n$ with a Liapunov boundary $\partial \Omega$. Denote

\begin{equation}
T_i f(z) = (-1)^i \int_{\Omega} H_i^*(x-z) f(x) dx^N
\end{equation}

where $H_i^*(x)$ is denoted as in (5), $i \in \mathbb{N}^*$, $f \in L^p(\Omega, C(V_{n,n}))$, $p \geq 1$. The operator $T_1$ is the Pompeiu operator $T$. Especially, we denote $f$ as $T_0 f$.

In [25], it is shown that, if $f \in L^p(\Omega, C(V_{n,n})), p \geq 1$, then $Tf \in C^\alpha(\Omega, C(V_{n,n})), \alpha = \frac{p-n}{p}$. $T_k f$ provides a particular weak solution to the inhomogeneous equation $D^k \omega = f(\text{weak})$ in $\Omega$. In this section, we shall show that, if $f \in \Lambda(f, \overline{\Omega})$, then $T_i f \in C^\infty(\Omega, C(V_{n,n})), i \in \mathbb{N}^*$ and $T_k f$ provides a particular solution to the inhomogeneous equation $D^k \omega = f$ in $\Omega$. 

\[
C_{j,0} = \begin{cases} 
1, & j = 0, \\
\frac{1}{2^{[\frac{j-1}{2}]}! \prod_{\mu=0}^{|\frac{j-1}{2}|} (n + 2\mu)}, & j \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}.
\end{cases}
\]
Theorem 1. Let $\Omega$ be an open non empty bounded subset of $\mathbb{R}^n$ with a Liapunov boundary $\partial \Omega$, $f \in \Lambda(f, \Omega)$. Then, for $z \in \Omega$

\begin{equation}
T_i f(z) = \sum_{j=0}^{\infty} (-1)^{j+i+j} \int_{\partial \Omega} H_{j+i+1}^*(x-z) d\sigma_x D^j f(x), \quad i \in \mathbb{N}.
\end{equation}

Proof. Step 1. For $f \in \Lambda(f, \Omega)$, we shall firstly prove

\begin{equation}
T_i f(z) = \sum_{j=0}^{k} (-1)^{j+i} \int_{\partial \Omega} H_{j+k+1}^*(x-z) d\sigma_x D^j f(x) + (-1)^{i+k+1} \int_{\Omega} H_{i+k+1}^*(x-z) D^{k+1} f(x) dx^N,
\end{equation}

where $i, k \in \mathbb{N}$, $z \in \Omega$. It is obvious that (11) is the direct result of Lemma 1 for $i = 0$.

For $i \geq 1$, in view of the properties of the kernel functions of $H_j^*(x-z)$

\begin{equation}
D \left[ H_{j+1}^*(x-z) \right] = \left[ H_{j+1}^*(x-z) \right] D = H_j^*(x-z), \quad x \in \mathbb{R}^n, \text{ for any } j \geq 1.
\end{equation}

Combining Stokes formulas with (12), we have

\begin{equation}
(-1)^i \int_{\Omega \setminus B(z, \epsilon)} H_i^*(x-z) f(x) dx^N = \sum_{j=0}^{k} (-1)^{j+i} \int_{\partial(\Omega \setminus B(z, \epsilon))} H_{j+i+1}^*(x-z) d\sigma_x D^j f(x) + (-1)^{i+k+1} \int_{\Omega \setminus B(z, \epsilon)} H_{i+k+1}^*(x-z) D^{k+1} f(x) dx^N.
\end{equation}

For $i \geq 1$ and $j \geq 0$, it is easy to check that,

\begin{equation}
\lim_{\epsilon \to 0} \int_{\partial B(z, \epsilon)} H_{j+i+1}^*(x-z) d\sigma_x D^j f(x) = 0.
\end{equation}
In view of the weak singularity of the kernel functions and (14), taking limits as $\varepsilon \to 0$ in (13), (11) holds.

Step 2. For $f \in \Lambda(f, \overline{\Omega})$, we shall show that

\[
\lim_{k \to \infty} \max_{z \in \Omega} \left| \int_{\Omega} H_{i+k+1}^* (x - z) D^{k+1} f(x) dx^N \right| = 0.
\]

Since $f \in \Lambda(f, \overline{\Omega})$, then there exist constants $C_0, M, 0 < C_0, M < +\infty$, and $N \in \mathbb{N}^*$, such that for any $k \geq N$

\[
\max_{x \in \Omega} |D^k f(x)| \leq C_0 M^k.
\]

Case 1. $n$ is odd. For any $k \geq N$, we have

\[
\int_{\Omega} H_{i+k+1}^* (x - z) D^{k+1} f(x) dx^N \leq 2^n A_{i+k+1} C_0 V(\Omega) M^{k+1} \delta^{i+k+1-n},
\]

where $\delta = \sup_{x_1, x_2 \in \Omega} \rho(x_1 - x_2)$, $V(\Omega)$ denotes the volume of $\Omega$. It is obvious that the series

\[
\sum_{k=1}^{\infty} 2^n A_{i+k+1} C_0 V(\Omega) M^{k+1} \delta^{i+k+1-n}
\]

converges. Then

\[
\lim_{k \to \infty} 2^n A_{i+k+1} C_0 V(\Omega) M^{k+1} \delta^{i+k+1-n} = 0,
\]

thus (15) holds.

Case 2. $n$ is even. In view of (5) and (7), it can be similarly proved that (15) holds.

Combining (11) with (15), taking limits $k \to \infty$ in (11), (10) follows.

By Theorem 1, we have
Corollary 1. Suppose that $f$ is $k$-regular in a domain $U$ in $\mathbb{R}^n$, $\Omega$ is an open non-empty bounded subset of $U$ with a Liapunov boundary $\partial \Omega$. Then, for $z \in \Omega$

$$(20) \quad T_i f(z) = \sum_{j=0}^{k-1} (-1)^{j+i} \int_{\partial \Omega} H^*_{j+i+1}(x-z) d\sigma_x D^j f(x), \quad i \in \mathbb{N}. $$

Remark 1. For $i = 0$, (20) is exactly the higher order Cauchy integral formula which has been obtained in [5, 24]. Analogous higher order Cauchy integral formula can be also found in [2, 3, 12].

Corollary 2. Let $\Omega$ be an open non-empty bounded subset of $\mathbb{R}^n$ with a Liapunov boundary $\partial \Omega$, $f \in \Lambda(f, \Omega)$. Then, for $z \in \Omega$

$$(21) \quad D[T_{i+1}f] = T_if, \quad i \in \mathbb{N}. $$

Remark 2. Corollary 2 implies that $T_k f$ provides a particular solution to the inhomogeneous equation $D^k \omega = f$ in $\Omega$ for $f \in \Lambda(f, \Omega)$. Especially, suppose $U$ is a domain in $\mathbb{R}^n$, $\Omega$ is an open non-empty bounded subset of $U$ with a Liapunov boundary $\partial \Omega$, $f$ is regular in $U$, then $T_k f$ is $(k+1)$-regular in $\Omega$. This result gives an improved result in [2, 25] under the assumption of $f \in \Lambda(f, \Omega)$.

Corollary 3. Let $U$ be a domain in $\mathbb{R}^n$, $\Omega$ be an open non-empty bounded subset of $U$ with a Liapunov boundary $\partial \Omega$, $f$ be a solution of equation $Lu = 0$ in $U$, where $Lu = Du + uh$, $h = \sum_{i=1}^{n} h_i e_i, h_i \in \mathbb{R}$ or $h$ be a real (complex) number. Then for $z \in \Omega$

$$(22) \quad T_i f(z) = \sum_{j=0}^{\infty} (-1)^{j+i} \int_{\partial \Omega} H^*_{j+i+1}(x-z) d\sigma_x D^j f(x), \quad i \in \mathbb{N}. $$
Proof. Obviously, if \( f \) is a solution of equation \( Lu = 0 \) in \( U \), where \( Lu = Du + uh \), \( h = \sum_{i=1}^{n} h_i e_i \) or \( h \) is a real (complex) number, then \( f \in \Lambda(f, \Omega) \). By Theorem 1, the result follows.

**Example 1.** Suppose \( u_i(x) = \sum_{k=0}^{\infty} \frac{(\alpha x_i e_i)^k}{k!} \triangleq e^{\alpha x_i e_i}, i = 1, \ldots, n \), where \( \alpha \) is a real number. Clearly, \( Du_i(x) = \alpha u_i(x) \). Thus for \( u_i(x), z \in \Omega \), by Corollary 3, (22) holds.

**Example 2.** Suppose \( h = \sum_{i=1}^{n} h_i e_i \), \( h_i \in \mathbb{R} \). Denote \( R = |h| = \sqrt{\sum_{i=1}^{n} h_i^2} \). Obviously, \( e^{Rx_i e_i} \) satisfies \( Du - Ru = 0 \), thus \( e^{Rx_i e_i} \) is also a solution of the Helmholtz equation \( \triangle u - R^2 u = 0 \). Then \( e^{Rx_i e_i} (R - h) \) is a solution of equation \( Du + uh = 0 \). For \( e^{Rx_i e_i} (R - h), z \in \Omega \), by Corollary 3, (22) holds.

\( \Omega \) is supposed to be an open non empty subset of \( \mathbb{R}^n \) with a Liapunov boundary \( \partial \Omega \). Denote

\[
\Pi f(z) = \begin{cases} 
\int_{\Omega} K(x - z) f(x) dx^N, & z \in \Omega, \\
\lim_{\xi \to z, \xi \in \Omega} \int_{\Omega} K(x - \xi) f(x) dx^N & z \in \partial \Omega,
\end{cases}
\]

where

\[
K(x) = \frac{1}{\omega_n} \left( \frac{(2 - n)e_1}{\rho^n(x)} - \frac{nx_1x}{\rho^{n+2}(x)} \right), \ x \in \mathbb{R}^n.
\]

\( f \in H^\alpha(\Omega, C(V_{n,n})), 0 < \alpha \leq 1 \), \( \Pi f \) is a singular integral to be taken in the Cauchy principal sense. In [25], we have proved the existence and Hölder continuity of \( \Pi f \) in \( \Omega \).

For \( u \in H^\alpha(\partial \Omega, C(V_{n,n})), 0 < \alpha \leq 1 \), denote

\[
(F_{\partial \Omega} u)(x) = \int_{\partial \Omega} H_1^*(y - x) d\sigma_y u(y), \ x \in \mathbb{R}^n \setminus \partial \Omega.
\]
(26) \((S_{\partial \Omega} u)(x) = \int_{\partial \Omega} H_1^*(y - x) d\sigma_y u(y), \ x \in \partial \Omega.\)

(27) \((F_{\partial \Omega}^+ u)(x) = \begin{cases} (F_{\partial \Omega} u)(x), & x \in \Omega^+, \\ \frac{1}{2} u(x) + (S_{\partial \Omega} u)(x) & x \in \partial \Omega. \end{cases}\)

**Theorem 2.** Let \(\Omega\) be an open non empty bounded subset of \(\mathbb{R}^n\) with a Liapunov boundary \(\partial \Omega\), \(f \in C^1(\Omega, C(V_{n,n}))\), \(\Pi f\) is defined as in (23). Then

(28) \(\Pi f(z) = (F_{\partial \Omega}^+ (\alpha e_1 \alpha f))(z) + T(e_1 D[f])(z) - \frac{2-n}{n} e_1 f(z), \ z \in \overline{\Omega},\)

where \(\alpha(x)\) denotes the unit outer normal of \(\partial \Omega\).

**Proof.** For \(z \in \Omega\), by Stokes formula, we have,

\[
\Pi f(z) = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(z,\varepsilon)} K(x-z) f(x) d\mathcal{X}^N
= \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(z,\varepsilon)} [H_1^*(x-z) e_1] Df(x) d\mathcal{X}^N
= \lim_{\varepsilon \to 0} \int_{\partial \Omega \setminus B(z,\varepsilon)} H_1^*(x-z) e_1 d\sigma_x f(x) + T(e_1 D[f])(z)
= \int_{\partial \Omega} H_1^*(x-z) e_1 d\sigma_x f(x) + T(e_1 D[f])(z) - \frac{2-n}{n} e_1 f(z).
\]

For \(z \in \partial \Omega\), taking limits in (29), (28) follows.

**Corollary 4.** Let \(\Omega\) be an open non empty bounded subset of \(\mathbb{R}^n\) with a Liapunov boundary \(\partial \Omega\), \(f \in \Lambda(f, \overline{\Omega})\), \(\Pi f\) is defined as in (23). Then in \(\Omega\)

(30) \(D[\Pi f] = e_1 D[f] + \frac{n-2}{n} D[e_1 f].\)
Corollary 5. Suppose that $f$ is regular in a domain $U$ in $\mathbb{R}^n$, $\Omega$ is an open non empty bounded subset of $U$ with a Liapunov boundary $\partial \Omega$. $\Pi f$ is defined as in (23). Then in $\Omega$

\begin{equation}
\triangle [\Pi f] = 0,
\end{equation}

where $\triangle$ is the Laplace operator.

3 Some applications

In this section, we shall give some applications of the higher order Cauchy-Pompeiu formula. The solutions of Dirichlet problems as well as the inhomogeneous equations $D^k u = f$ are obtained. In the sequel, $K_n$ denotes the unit ball in $\mathbb{R}^n$ ($n \geq 3$), more clearly,

$$K_n = \{x|x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, |x| < 1\}.$$ 

Denote

\begin{equation}
G(y, x) = \frac{1}{\rho^{n-2}(y-x)} - \frac{1}{|y|^{n-2}\rho^{n-2}(\frac{y}{|y|^2} - x)}, x \in K_n, y \in K_n, x \neq y.
\end{equation}

Remark 3. $G(y, x)$ has the following properties:

1. $\triangle_x G(y, x) = 0, \ x \in K_n \ \backslash \ {y}$.
2. $G(y, x) = G(x, y), \ x, y \in K_n, x \neq y$.
3. $G(y, x) = 0, \ y \in \partial K_n, x \in K_n$.

Theorem 3. Suppose $f \in C^2(\overline{K_n}, C(V_n, n))$, then for $x \in K_n$

\begin{equation}
f(x) = \frac{1}{\omega_n} \int_{\partial K_n} \frac{1 - |x|^2}{\rho^n(y-x)} f(y) dS_y + \frac{1}{(2-n)\omega_n} \int_{K_n} G(y, x) \triangle y f(y) dy^N.
\end{equation}
Proof. By Lemma 1, for \( x \in K_n \), we have

\[
(34)\quad f(x) = \frac{1}{\omega_n} \int_{\partial K_n} \frac{y-x}{\rho^n(y-x)} \, \mathrm{d}\sigma_y f(y) - \frac{1}{(2-n)\omega_n} \int_{\partial K_n} \frac{1}{\rho^{n-2}(y-x)} \, \mathrm{d}\sigma_y D[f](y) \\
+ \frac{1}{(2-n)\omega_n} \int_{K_n} \frac{1}{\rho^{n-2}(y-x)} \Delta_y f(y) \, \mathrm{d}y^N.
\]

By Stokes formula, for \( x \in K_n \) and \( x \neq 0 \), we have

\[
(35)\quad 0 = \frac{1}{\omega_n} \int_{\partial K_n} \frac{y-x}{|x|^2} \, \rho^n(y-x) \, \rho^n(y-x) \, \mathrm{d}\sigma_y f(y) - \frac{1}{(2-n)\omega_n} \int_{\partial K_n} \frac{1}{\rho^{n-2}(y-x)} \, \mathrm{d}\sigma_y D[f](y) \\
+ \frac{1}{(2-n)\omega_n} \int_{K_n} \frac{1}{\rho^{n-2}(y-x)} \, \rho^{n-2}(y-x) \Delta_y f(y) \, \mathrm{d}y^N.
\]

(35) can be rewritten as

\[
(36)\quad 0 = \frac{1}{\omega_n} \int_{\partial K_n} |x|^2 \left( \frac{y-x}{|x|^2} \right) \, \rho^n(y-x) \, \mathrm{d}\sigma_y f(y) - \\
- \frac{1}{(2-n)\omega_n} \int_{\partial K_n} |x|^{n-2} \rho^{n-2}(y-x) \, \rho^{n-2}(y-x) \, \mathrm{d}\sigma_y D[f](y) + \\
+ \frac{1}{(2-n)\omega_n} \int_{K_n} |x|^{n-2} \rho^{n-2}(y-x) \, \rho^{n-2}(y-x) \Delta_y f(y) \, \mathrm{d}y^N.
\]

In view of

\[
(37)\quad |x|^k \rho^k(y-x) = |y|^k \rho^k \left( \frac{y}{|y|^2} - x \right), \quad k \in \mathbb{N}^*,
\]

combining (34), (36) with (37), (33) follows.

For \( x = 0 \), by Stokes formula and (34), (33) still holds. Thus the result is proved.
Remark 4. Suppose \( f \in C^2(\overline{K_n}, C(V_{n,n})) \), moreover, \( f \) is harmonic in \( K_n \). Then for \( x \in K_n \)

\[
f(x) = \frac{1}{\omega_n} \int_{\partial K_n} \frac{1 - |x|^2}{\rho^n(y - x)} f(y) dS_y.
\]

(38) is exactly the Poisson expression of harmonic functions.

Theorem 4. The solution of the Dirichlet problem for the Poisson equation in the unit ball \( K_n \)

\[
\Delta u = f \text{ in } K_n, \quad u = \gamma \text{ on } \partial K_n,
\]

for \( f \in \Lambda(f, K_n) \) and \( \gamma \in C(\partial K_n, C(V_{n,n})) \) is uniquely given by

\[
u(x) = \frac{1}{\omega_n} \int_{\partial K_n} \frac{1 - |x|^2}{\rho^n(y - x)} \gamma(y) dS_y + \frac{1}{(2 - n)\omega_n} \int_{K_n} G(y, x) f(y) dy^N.
\]

(39) Proof. It can be directly proved by Corollary 2, Theorem 3, Remark 3 and Remark 4.

Lemma 2. (see [26]) If \( f \) is \( k \)-regular in an open neighborhood \( \Omega \) of the origin, then in a suitable open ball \( B(0, R) \subset \Omega \)

\[
f(x^\nu) = f(0) + \sum_{p=1}^{\infty} \sum_{j=0}^{k-1} \sum_{(l_1, \ldots, l_{p-j})} C_{j, p-j} x^l V_{l_1, \ldots, l_{p-j}}(x^\nu) C_{l_1, \ldots, l_{p-j}}.
\]

\( C_{j, p-j} \) and \( C_{l_1, \ldots, l_{p-j}} \) are constants which are suitably chosen.

By Lemma 2 and Corollary 2, we have

Theorem 5. The solutions of inhomogeneous equations in the unit ball \( K_n \)

\[
D^k u = f \text{ in } K_n,
\]
for $f \in \Lambda(f, K_n)$ are given in a suitable open ball $\overset{\circ}{B}(0,R) \subset K_n$ by

$$
(41) \quad u = C_0 + \sum_{p=1}^{\infty} \sum_{j=0}^{k-1} \sum_{l_1, \ldots, l_{p-j}} C_{j,p-j} x^j V_{l_1, \ldots, l_{p-j}}(x^n) C_{l_1, \ldots, l_{p-j}} + T_k f.
$$

$C_0$, $C_{j,p-j}$ and $C_{l_1, \ldots, l_{p-j}}$ are constants which are suitably chosen.

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**References**


School of Mathematics and Statistics, 
Wuhan University, 
Wuhan 430072, P. R. China 

E–mail: zhangzx9@sohu.com