Certain Divisible Hypergroups

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

A group $G$ is said to be divisible if for every $x \in G$ and every $n \in \mathbb{N}$, $x = y^n$ for some $y \in G$ where $\mathbb{N}$ is the set of all positive integers. More generally, we call a hypergroup $(A, \circ)$ a divisible hypergroup if for every $x \in A$ and every $n \in \mathbb{N}$, $x \in (y, \circ)^n$ for some $y \in A$ where $(y, \circ)^n$ denotes $y \circ y \circ \ldots \circ y$ ($n$ copies). If $G$ is any group and $H < G$, let $G/H$ and $G|H$ be respectively the sets $\{xH | x \in G\}$ and $\{HxH | x \in G\}$. It is known that $(G/H, \circ)$ and $(G|H, \circ)$ are hypergroups where $xH \circ yH = \{tH | t \in xHy\}$ and $HxH \circ H\circ H = \{HtH | t \in xHy\}$. These hypergroups will be shown to be divisible if the group $G$ is divisible. Let $U_n(\mathbb{R})$ be the group under multiplication of all nonsingular upper triangular $n \times n$ matrices over $\mathbb{R}$. Then the group $U_n(\mathbb{R})$ is not divisible. However, it is known that the group $U_n^+(\mathbb{R}) = \{A \in U_n(\mathbb{R}) | A_{ii} > 0 \text{ for all } i \in \{1, \ldots, n\}\}$ is divisible. Based on this result, we show that there are infinitely many subgroups $H$ of $U_n(\mathbb{R})$ such that the hypergroups $(U_n(\mathbb{R})/H, \circ)$ and $(U_n(\mathbb{R})|H, \circ)$ are divisible.

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1 Introduction

The cardinality of a set $X$ will be denoted by $|X|$. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote respectively the set of positive integers, the set of integers, the set of rational numbers and the set of real numbers. For any subfield $F$ of the field $\mathbb{R}$, let $F^* = F\setminus \{0\}$ and $F^+ = \{x \in F | x > 0\}$.

We call a group $G$ a divisible group if for every $x \in G$ and every $n \in \mathbb{N}$, $x = y^n$ for some $y \in G$. The the additive group $(\mathbb{Q}, +)$ is divisible while the multiplicative group $(\mathbb{Q}^+, \cdot)$ is not divisible. The group $(\mathbb{R}^+, \cdot)$ is clearly divisible. Divisible abelian groups have been characterized in terms of $\mathbb{Z}$-injectively. This can be seen in [2], page 195. It is also known that every nonzero finite abelian group is not divisible ([2], page 198). In fact, a more general result is obtained from [5] as follows:

**Proposition 1.** ([5]) If $G$ is a nontrivial finite group, then $G$ is not divisible.

Let $M_n(\mathbb{R})$ be the semigroup of all $n \times n$ matrices over $\mathbb{R}$ under matrix multiplication. Then the unit group of the semigroup $M_n(\mathbb{R})$ is

$$G_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) | \det A \neq 0 \}$$

For each $A \in M_n(\mathbb{R})$, the entry of $A$ in the $i^{th}$ row and the $j^{th}$ column will be denoted by $A_{i,j}$. Next, let

$$U_n(\mathbb{R}) = \{ A \in G_n(\mathbb{R}) | A \text{ is upper triangular} \}.$$

Then $U_n(\mathbb{R})$ is a subgroup of $G_n(\mathbb{R})$ ([3], page 410). For convenience, let

$$U_n^+(\mathbb{R}) = \{ A \in G_n(\mathbb{R}) | A_{ii} > 0 \text{ for all } i \in \{1, ..., n\} \}.$$

If $A, B \in U_n^+(\mathbb{R})$, then for every $i \in \{1, ..., n\}$, $(AB)_{ii} = A_{ii}B_{ii} > 0$ and $(A^{-1})_{ii} = \frac{1}{A_{ii}} > 0$, so $U_n^+(\mathbb{R})$ is a subgroup of $U_n(\mathbb{R})$ and $G_n(\mathbb{R})$. The
groups $G_n(\mathbb{R})$ and $U_n(\mathbb{R})$ are clearly not divisible. An interesting result for the group $U_n^+(\mathbb{R})$ was given by N. Triphop and A. Wasanawichit [4] as follows:

**Theorem 1.** ([4]) For every $n \in \mathbb{N}$, $U_n^+(\mathbb{R})$ is a divisible group.

The notation of divisibility is defined more extensively in this paper. Divisible hypergroups will be defined. Let us recall some hyperstructures which will be used. A hyperoperation on a nonempty set $A$ is a mapping $\circ : A \times A \to P^*(A)$ where $P(A)$ is the power set of $A$ and $P^*(A) = P(A) \setminus \{\emptyset\}$, and $(A, \circ)$ is called a hypergroupoid. If $X$ and $Y$ are nonempty subsets of $A$, let

$$X \circ Y = \bigcup_{x \in X, y \in Y} (x \circ y).$$

A semihypergroups is a hypergroupoid $(A, \circ)$ such that $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in A$. A semihypergroup $(A, \circ)$ with $A \circ x = x \circ A = A$ for all $x \in A$ is called a hypergroup. A hypergroup $(A, \circ)$ is said to be divisible if for any $x \in A$ and every $n \in \mathbb{N}$, $x \in (y, \circ)^n$ for some $y \in A$ where $(y, \circ)^n$ denotes the set $y \circ y \circ \ldots \circ y$ ($n$ copies). Then a total hypergroup, that is, a hypergroup $(A, \circ)$ with $x \circ y = A$ for all $x, y \in A$, is clearly divisible.

Let $G$ be a group and $H$ a subgroup of $G$. It is well-known that the relation $\sim$ defined on $G$ by $a \sim b \iff a = bx$ for some $x \in H$ is an equivalence relation on $G$ and the $\sim$-class of $G$ containing $a \in G$ is $aH$ and $aH = H$ $\iff a \in H$. Similarly, it is easy to verify the relation $\approx$ defined on $G$ by $a \approx b \iff a = xby$ for some $x, y \in H$ is an equivalence relation on $G$ and the $\approx$-class of $G$ containing $a \in G$ is $HaH$. Moreover, $HaH = H$ $\iff a \in H$. The notation $G/H$ denotes the set of all left cosets of $H$ in $G$, that is,

$$G/H = \{xH \mid x \in G\}.$$
Define the hyperoperation $\circ$ on $G/H$ by

$$xH \circ yH = \{tH \mid t \in xHy\} \text{ for all } x, y \in G.$$ 

Also, let $G\mid H$ and $\diamond$ the hyperoperation defined on $G\mid H$ as follows:

$$G\mid H = \{HxH \mid x \in G\},$$

$$HxH \diamond HyH = \{HtH \mid t \in xHy\} \text{ for all } x, y \in G.$$ 

Then $(G/H, \circ)$ and $(G\mid H, \diamond)$ are both hypergroups ([1], page 11). Notice that if $H$ is normal in $G$, then $(G/H, \circ) = (G\mid H, \diamond)$ which is the quotient group of $G$ by $H$. Moreover, if $H_1$ and $H_2$ are subgroups of $G$ such that $H_1 \neq H_2$, then $G/H_1 \neq G/H_2$ and $G\mid H_1 \neq G\mid H_2$.

Our main purpose is to show that there are infinite many subgroups $H$ of $U_n(\mathbb{R})$ such that the hypergroups $(U_n(\mathbb{R})/H, \circ)$ and $(U_n(\mathbb{R})\mid H, \diamond)$ are divisible. Theorem 1 is helpful for our work.

2 Basic Properties

Throughout this section, let $G$ be any group, $H$ a subgroup of $G$. Also, $(G/H, \circ)$ and $(G\mid H, \diamond)$ are hypergroups defined previously.

Lemma 1. For $x \in G$ and $n \in \mathbb{N}\setminus\{1\}$,

$$(xH, \circ)^n = \{tH \mid t \in (xH)^{n-1}x\}$$

and

$$(HxH, \circ)^n = \{HtH \mid t \in (xH)^{n-1}x\}$$

Hence $x^nH \in (xH, \circ)^n$ and $Hx^nH \in (HxH, \diamond)^n$ for all $n \in \mathbb{N}$. In particular, $(H, \circ)^n = \{H\} = (H, \diamond)^n$. 

Proof. This is clear for \( n = 2 \). If \( k \geq 2 \) is such that \((xH, \circ)^{k} = \{tH \mid t \in (xH)^{k-1}x\}\) and \((HxH, \circ)^{k} = \{tH \mid t \in (xH)^{k-1}x\}\). Hence

\[
(xH, \circ)^{k+1} = xH \circ \{tH \mid t \in (xH)^{k-1}x\} = \{rH \mid r \in xHt \text{ for some } t \in (xH)^{k-1}x\} = \{tH \mid t \in (xH)^{k}x\},
\]

and

\[
(HxH, \circ)^{k+1} = HxH \circ \{HtH \mid t \in (xH)^{k-1}x\} = \{HrH \mid r \in xHt \text{ for some } t \in (xH)^{k-1}x\} = \{HtH \mid t \in (xH)^{k}x\}.
\]

If \( x, y \in G \) and \( n \in \mathbb{N} \) are such that \( x = y^{n} \), then \( xH = y^{n}H \in (yH, \circ)^{n} \) and \( HxH = Hy^{n}H \in (HyH, \circ)^{n} \) by Lemma 1. Hence we have:

**Proposition 2.** If \( G \) is a divisible group, then both \((G/H, \circ)\) and \((G|H, \circ)\) are divisible hypergroups.

For any group \( G \) if \( H = G \), then \(|G/H| = 1 = |G|\), so \((G/H, \circ)\) and \((G|H, \circ)\) are divisible hypergroups. Hence the converse of Proposition 2 is not generally true. A nontrivial example is as follows:

**Example 1** By Proposition 1, \( S_{3} \) is not a divisible group. Let \( H \) be the subgroup of \( S_{3} \) generated by the cycle \((1 2)\), that is, \( H = \{(1), (1 2)\} \). Since \(|S_{3}/H| = \frac{6}{2} = 3\) and \((1 3)^{-1}(2 3) = (1 3)(2 3) = (1 2 3) \notin H\), it follows that \( H \notin (1 3)H, (2 3)H \notin H\). Thus

\[S_{3}/H = \{H, (1 3)H, (2 3)H\}.\]

Since \((1 3) \notin H, (2 3) \notin H\),

\[(1 3) \in H(1 3)H = (1 3)H \cup (1 2)(1 3)H = \]
\[ = (1 3)H \cup (1 2 3)H = (1 3)H \cup (2 3)H \text{ since } (1 2 3) = (2 3)(1 2) \]

and

\[ S_3 = H \cup (1 3)H \cup (2 3)H, \]

it follows that

\[ S_3|H = \{ H, H(1 3)H \} \text{ and } H(2 3)H = H(1 3)H. \]

We know that \( (1 3) = (2 3)(1 2)(2 3) \in (2 3)H(2 3) \) and \( (1 3) = (1 3)^3. \)

By Lemma 2,

\[ (1 3)H \in ((1 2 3)H, \circ)^3, \]

\[ (1 3)H = (1 3)^3H \in ((1 3)H, \circ)^3, \]

\[ H(1 3)H \in (H(2 3)H, \circ)^2 = (H(1 3)H, \circ)^2, \]

\[ H(1 3)H = H(1 3)^3H \in (H(1 3)H, \circ)^3. \]

Next, let \( n \in \mathbb{N} \) be such that \( n \geq 3. \) If \( n \) is odd, then \( (1 3) = (1 3)^n, \) so by Lemma 2

\[ (1 3)H = (1 3)^nH \in ((1 3)H, \circ)^n \]

and

\[ H(1 3)H = H(1 3)H = H(1 3)^nH \in (H(1 3)H, \circ)^n. \]

If \( n \) is even, then

\[ (1 3) = (2 3)^{n-2}(2 3)(1 2)(2 3) \in ((2 3)H)^{n-2}(2 3)H(2 3) = ((2 3)H, \circ)^{n-1}(2 3), \]

thus by Lemma 2,

\[ (1 3)H \in ((2 3)H, \circ)^n \]

and

\[ H(1 3)H \in (H(2 3)H, \circ)^n = (H(1 3)H, \circ)^n. \]

This shows that for every \( n \in \mathbb{N}, \) \( (1 3)H \in ((1 3)H, \circ)^n \) or \( (1 3)H \in ((2 3)H, \circ)^n \) and \( H(1 3)H \in (H(1 3)H, \circ)^n. \) We can show similarly that for every \( n \in \mathbb{N}, \) \( (2 3)H \in ((2 3)H, \circ)^n \) or \( (2 3)H \in ((1 3)H, \circ)^n. \)
Hence we have that $S_3$ is not a divisible group and $H \neq S_3$, but $(S_3/H, \circ)$ and $(S_3|H, \circ)$ are divisible hypergroups.

3 The Hypergroups $(U_n(\mathbb{R})/H, \circ)$ and $(U_n(\mathbb{R})|H, \diamond)$

For each prime $p$, $\mathbb{Q}(\sqrt{p})$ is a subfield of $\mathbb{R}$ and if $p_1$ and $p_2$ are distinct primes, then $\mathbb{Q}(\sqrt{p_1}) \neq \mathbb{Q}(\sqrt{p_2})$. Hence there are infinitely many subfields of $\mathbb{R}$. For each subfield $F$ of $\mathbb{R}$, let

$$H_F = \{ A \in U_n(\mathbb{R}) | A_{ii} \in F^* \text{ for all } i \in \{1, \ldots, n\} \}.$$

Clearly, for distinct $F_1$ and $F_2$ of $\mathbb{R}$, $H_{F_1} \neq H_{F_2}$.

**Lemma 2.** For every subfield $F$ of $\mathbb{R}$, $H_F$ is a subgroup of the group $U_n(\mathbb{R})$.

**Proof.** Since for $A, B \in H_F$, $(AB)_{ii} = A_{ii}B_{ii} \in F^*$ and $(A^{-1})_{ii} = \frac{1}{A_{ii}} \in F^*$ for all $i \in \{1, \ldots, n\}$, it follows that $H_F$ is a subgroup of $U_n(\mathbb{R})$.

**Theorem 2.** If $F$ is a subfield of $\mathbb{R}$, then $(U_n(\mathbb{R})/H_F, \circ)$ and $(U_n(\mathbb{R})|H_F, \diamond)$ are both divisible hypergroups.

**Proof.** Let $A \in U_n(\mathbb{R})$ and $m \in \mathbb{N}$. Define the diagonal matrix $B \in U_n(\mathbb{R})$ by $B_{ii} = 1$ if $A_{ii} = -1$ if $A_{ii} < 0$. Then $B$ is clearly an element of $H_F$ and $AB = C^m$. Thus $A = C^mB^{-1}$ and hence $AH_F = C^mB^{-1}H_F = C^mH_F$ and $H_FAH_F = H_FC^mB^{-1}H_F = H_FC^mH_F$. But $C^mH_F \in (CH_F, \circ)^m$ and $H_FC^mH_F \in (H_FC^mH_F, \circ)^m$ by Lemma 1, so $AH_F \in (CH_F, \circ)^m$ and $H_FAH_F \in (H_FC^mH_F, \circ)^m$.

Hence the theorem is proved.

**Remark 1.** If $F_1$ and $F_2$ are distinct subfields of $\mathbb{R}$, then $H_{F_1} \neq H_{F_2}$ which implies that $U_n(\mathbb{R})/H_{F_1} \neq U_n(\mathbb{R})/H_{F_2}$ and $U_n(\mathbb{R})|H_{F_1} \neq U_n(\mathbb{R})|H_{F_2}$. 
References


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