On a subclass of $n$-close to convex functions associated with some hyperbola

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Dedicated to Professor DumitruAcu on his 60th anniversary

Abstract

In this paper we define a subclass of $n$-close to convex functions associated with some hyperbola and we obtain some properties regarding this class.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

We recall here the definition of the well-known class of close to convex functions:

$$CC = \left\{ f \in A : \text{exists } g \in S^* , \ Re \frac{zf'(z)}{g(z)} > 0 , z \in U \right\} .$$
Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

\begin{equation}
(1) \quad f(z) = L_a F(z) = \frac{1 + a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt , \quad a \in \mathbb{C}, \quad \text{Re } a \geq 0.
\end{equation}

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the case $a = 1, 2, 3, \ldots$ we obtain the Bernardi integral operator.

Let $D^n$ be the S'al'agean differential operator (see [6]) $D^n : A \to A, n \in \mathbb{N}$, defined as:

\begin{align*}
D^0 f(z) &= f(z) \\
D^1 f(z) &= Df(z) = zf'(z) \\
D^n f(z) &= D(D^{n-1} f(z))
\end{align*}

We observe that if $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$ then $D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$.

The purpose of this note is to define a subclass of $n$-close to convex functions associated with some hyperbola and to obtain some estimations for the coefficients of the series expansion and some other properties regarding this class.

### 2 Preliminary results

**Definition 1.** (see [7]) A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies

\[
\left| \frac{zf'(z)}{f(z)} - 2\alpha \left( \sqrt{2} - 1 \right) \right| < \text{Re} \left\{ \sqrt{2} \frac{zf'(z)}{f(z)} \right\} + 2\alpha \left( \sqrt{2} - 1 \right),
\]

for some $\alpha (\alpha > 0)$ and for all $z \in U$. 

Definition 2. (see [2]) Let \( f \in S \) and \( \alpha > 0 \). We say that the function \( f \) is in the class \( SH_\alpha(\alpha) \), \( n \in \mathbb{N} \), if
\[
\left| \frac{D^{n+1}f(z)}{D^n f(z)} - 2\alpha \left( \sqrt{2} - 1 \right) \right| < \text{Re} \left\{ \frac{\sqrt{2}}{2} \frac{D^{n+1}f(z)}{D^n f(z)} \right\} + 2\alpha \left( \sqrt{2} - 1 \right) , \quad z \in U .
\]

Remark 1. Geometric interpretation: If we denote with \( p_\alpha \) the analytic and univalent functions with the properties \( p_\alpha(0) = 1 \), \( p_\alpha'(0) > 0 \) and \( p_\alpha(U) = \Omega(\alpha) \), where \( \Omega(\alpha) = \{ w = u + i \cdot v : v^2 < 4\alpha u + u^2 , \ u > 0 \} \) (note that \( \Omega(\alpha) \) is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin), then \( f \in SH_\alpha(\alpha) \) if and only if \( \frac{D^{n+1}f(z)}{D^n f(z)} < p_\alpha(z) \), where the symbol \(<\) denotes the subordination in \( U \). We have \( p_\alpha(z) = (1 + 2\alpha)\sqrt{\frac{1 + bz}{1 - z}} - 2\alpha \), \( b = b(\alpha) = \frac{1 + 4\alpha - 4\alpha^2}{(1 + 2\alpha)^2} \) and the branch of the square root \( \sqrt{w} \) is chosen so that \( \text{Im} \sqrt{w} \geq 0 \). If we consider \( p_\alpha(z) = 1 + C_1 z + \ldots \), we have \( C_1 = \frac{1 + 4\alpha}{1 + 2\alpha} \).

Remark 2. If we denote by \( D^n g(z) = G(z) \), we have: \( g \in SH_\alpha(\alpha) \) if and only if \( G \in SH(\alpha) = SH_0(\alpha) \).

Theorem 1. (see [2]) If \( F(z) \in SH_\alpha(\alpha) , \ \alpha > 0 , \ n \in \mathbb{N} , \) and \( f(z) = L_\alpha F(z) \), where \( L_\alpha \) is the integral operator defined by (1), then \( f(z) \in SH_\alpha(\alpha) , \ \alpha > 0 , \ n \in \mathbb{N} \).

Definition 3. (see [1]) Let \( f \in A \) and \( \alpha > 0 \). We say that the function \( f \) is in the class \( CCH(\alpha) \) with respect to the function \( g \in SH(\alpha) \) if
\[
\left| \frac{zf'(z)}{g(z)} - 2\alpha \left( \sqrt{2} - 1 \right) \right| < \text{Re} \left\{ \frac{\sqrt{2}}{2} \frac{zf'(z)}{g(z)} \right\} + 2\alpha \left( \sqrt{2} - 1 \right) , \quad z \in U .
\]

Remark 3. Geometric interpretation: \( f \in CCH(\alpha) \) with respect to the function \( g \in SH(\alpha) \) if and only if \( \frac{zf'(z)}{g(z)} \) take all values in the convex domain \( \Omega(\alpha) \), where \( \Omega(\alpha) \) is defined in Remark 1.
Theorem 2. (see [1]) If \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) belong to the class \( CCH(\alpha) \), \( \alpha > 0 \), with respect to the function \( g(z) \in SH(\alpha) \), \( \alpha > 0 \), \( g(z) = z + \sum_{j=2}^{\infty} b_j z^j \), then
\[
|a_2| \leq \frac{1 + 4\alpha}{1 + 2\alpha}, \quad |a_3| \leq \frac{(1 + 4\alpha)(11 + 56\alpha + 72\alpha^2)}{12(1 + 2\alpha)^3}.
\]

The next theorem is result of the so called "admissible functions method" due to P.T. Mocanu and S.S. Miller (see [3], [4], [5]).

Theorem 3. Let \( q \) be convex in \( U \) and \( j : U \to \mathbb{C} \) with \( \text{Re}[j(z)] > 0 \), \( z \in U \). If \( p \in \mathcal{H}(U) \) and satisfied \( p(z) + j(z) \cdot zp'(z) \prec q(z) \), then \( p(z) \prec q(z) \).

3 Main results

Definition 4. Let \( f \in A, n \in \mathbb{N} \) and \( \alpha > 0 \). We say that the function \( f \) is in the class \( CCH_n(\alpha) \), with respect to the function \( g \in SH_n(\alpha) \), if
\[
\left| \frac{D^{n+1}f(z)}{D^ng(z)} - 2\alpha \left( \sqrt{2} - 1 \right) \right| < \text{Re} \left\{ \sqrt{2} \frac{D^{n+1}f(z)}{D^ng(z)} + 2\alpha \left( \sqrt{2} - 1 \right) \right\}, \quad z \in U.
\]

Remark 4. Geometric interpretation: \( f \in CCH_n(\alpha) \), with respect to the function \( g \in SH_n(\alpha) \), if and only if \( \frac{D^{n+1}f(z)}{D^ng(z)} \prec p_\alpha(z) \), where the symbol \( \prec \) denotes the subordination in \( U \) and \( p_\alpha \) is defined in Remark 1.

Remark 5. If we denote \( D^n f(z) = F(z) \) and \( D^n g(z) = G(z) \) we have:
\( f \in CCH_n(\alpha) \), with respect to the function \( g \in SH_n(\alpha) \), if and only if \( F \in CCH(\alpha) \), with respect to the function \( G \in SH(\alpha) \) (see Remark 2).

Theorem 4. Let \( \alpha > 0 \), \( n \in \mathbb{N} \) and \( f \in CCH_n(\alpha) \), \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \), with respect to the function \( g \in SH_n(\alpha) \), then
\[
|a_2| \leq \frac{1}{2^n} \cdot \frac{1 + 4\alpha}{1 + 2\alpha}, \quad |a_3| \leq \frac{1}{3^n} \cdot \frac{(1 + 4\alpha)(11 + 56\alpha + 72\alpha^2)}{12(1 + 2\alpha)^3}.
\]
Proof. If we denote by $D^n f(z) = F(z)$, $F(z) = \sum_{j=2}^{\infty} b_j z^j$, we have (using Remark 5) from the above series expansions we obtain $|a_j| \leq \frac{1}{j^n} \cdot |b_j|$, $j \geq 2$. Using the estimations from the Theorem 2 we obtain the needed results.

**Theorem 5.** Let $\alpha > 0$ and $n \in \mathbb{N}$. If $F(z) \in CCH_n(\alpha)$, with respect to the function $G(z) \in SH_n(\alpha)$, and $f(z) = L_\alpha F(z)$, $g(z) = L_\alpha G(z)$, where $L_\alpha$ is the integral operator defined by (1), then $f(z) \in CCH_n(\alpha)$, with respect to the function $g(z) \in SH_n(\alpha)$.

**Proof.** By differentiating (1) we obtain $(1 + a) F(z) = a f(z) + z f'(z)$ and $(1 + a) G(z) = a g(z) + z g'(z)$.

By means of the application of the linear operator $D^{n+1}$ we obtain

$$(1 + a) D^{n+1} F(z) = a D^{n+1} f(z) + D^{n+1} (zf'(z))$$

or

$$(1 + a) D^{n+1} F(z) = a D^{n+1} f(z) + D^{n+2} f(z)$$

Similarly, by means of the application of the linear operator $D^n$ we obtain

$$(1 + a) D^n G(z) = a D^n g(z) + D^{n+1} g(z)$$

Thus

$$\frac{D^{n+1} F(z)}{D^n G(z)} = \frac{D^{n+2} f(z) + a D^{n+1} f(z)}{D^{n+1} g(z) + a D^n g(z)} =$$

$$= \frac{D^{n+2} f(z)}{D^{n+1} g(z)} \cdot \frac{D^{n+1} g(z)}{D^n g(z)} + a \cdot \frac{D^{n+1} f(z)}{D^n g(z)}$$

(2)

With notations $D^{n+1} f(z) = p(z)$ and $D^{n+1} g(z) = h(z)$, by simple calculations, we have

$$\frac{D^{n+2} f(z)}{D^{n+1} g(z)} = p(z) + \frac{1}{h(z)} \cdot z p'(z)$$
Thus from (2) we obtain

$$\frac{D^{n+1}F(z)}{D^nG(z)} = \frac{h(z) \cdot \left( zp'(z) \cdot \frac{1}{h(z)} + p(z) \right) + a \cdot p(z)}{h(z) + a} = p(z) + \frac{1}{h(z) + a} \cdot zp'(z)$$

(3)

From Remark 4 we have \( \frac{D^{n+1}F(z)}{D^nG(z)} \prec p_\alpha(z) \) and thus, using (3), we obtain

$$p(z) + \frac{1}{h(z) + a} zp'(z) \prec p_\alpha(z).$$

We have from Remark 1 and from the hypothesis \( Re \frac{1}{h(z) + a} > 0 \), \( z \in U \).

In this conditions from Theorem 3 we obtain \( p(z) \prec p_\alpha(z) \) or \( \frac{D^{n+1}f(z)}{D^n\gamma(z)} \prec p_\alpha(z) \). This means that \( f(z) = L_\alpha F(z) \in CCH_n(\alpha) \), with respect to the function \( g(z) = L_\alpha G(z) \in SH_n(\alpha) \) (see Theorem 1).

**Theorem 6.** Let \( a \in \mathbb{C} \), \( Re a \geq 0 \), \( \alpha > 0 \), and \( n \in \mathbb{N} \). If \( F(z) \in CCH_n(\alpha) \), with respect to the function \( G(z) \in SH_n(\alpha) \), \( F(z) = z + \sum_{j=2}^{\infty} a_j z^j \), and \( g(z) = L_\alpha G(z) \), \( f(z) = L_\alpha F(z) \), \( f(z) = z + \sum_{j=2}^{\infty} b_j z^j \), where \( L_\alpha \) is the integral operator defined by (1), then

$$|b_2| \leq \left| a + 1 \right| \frac{1}{a + 2} \cdot \frac{1 + 4\alpha}{1 + 2\alpha}, \quad |b_3| \leq \left| a + 1 \right| \frac{1}{a + 3} \cdot \frac{1}{3^n} \cdot \frac{(1 + 4\alpha)(11 + 56\alpha + 72\alpha^2)}{12(1 + 2\alpha)^3}.$$

**Proof.** From \( f(z) = L_\alpha F(z) \) we have \( (1 + a)F(z) = af(z) + zf'(z) \). Using the above series expansions we obtain

$$z + \sum_{j=2}^{\infty} (1 + a) a_j z^j = az + \sum_{j=2}^{\infty} ab_j z^j + z + \sum_{j=2}^{\infty} jb_j z^j$$
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and thus \( b_j(a + j) = (1 + a)a_j, \ j \geq 2 \). From the above we have \( |b_j| \leq \frac{a + 1}{a + j} \cdot |a_j|, \ j \geq 2 \). Using the estimations from Theorem 4 we obtain the needed results.

For \( a = 1 \), when the integral operator \( L_a \) become the Libera integral operator, we obtain from the above theorem:

**Corollary 1.** Let \( \alpha > 0 \) and \( n \in \mathbb{N} \). If \( F(z) \in CCH_n(\alpha) \), with respect to the function \( G(z) \in SH_n(\alpha) \), \( F(z) = z + \sum_{j=2}^{\infty} a_j z^j \), and \( g(z) = L(G(z)) \), \( f(z) = L(F(z)) \), \( f(z) = z + \sum_{j=2}^{\infty} b_j z^j \), where \( L \) is Libera integral operator defined by \( L(H(z)) = 2z \int_{0}^{z} H(t)dt \), then

\[
|b_2| \leq \frac{1}{2^{n-1}} \cdot \frac{1 + 4\alpha}{3 + 6\alpha}, \ |b_3| \leq \frac{1}{3^n} \cdot \frac{(1 + 4\alpha)(11 + 56\alpha + 72\alpha^2)}{24(1 + 2\alpha)^3}.
\]

**Theorem 7.** Let \( n \in \mathbb{N} \) and \( \alpha > 0 \). If \( f \in CCH_{n+1}(\alpha) \) then \( f \in CCH_n(\alpha) \).

**Proof.** With notations \( \frac{D^{n+1}f(z)}{D^ng(z)} = p(z) \) and \( \frac{D^{n+1}g(z)}{D^ng(z)} = h(z) \) we have (see the proof of the Theorem 5):

\[
\frac{D^{n+2}f(z)}{D^{n+1}g(z)} = p(z) + \frac{1}{h(z)} \cdot zp'(z).
\]

From \( f \in CCH_{n+1}(\alpha) \) we obtain (see Remark 4) \( p(z) + \frac{1}{h(z)} \cdot zp'(z) \prec p_\alpha(z). \) Using the Remark 1 we have \( Re\frac{1}{h(z)} > 0, \ z \in U \), and from Theorem 3 we obtain \( p(z) \prec p_\alpha(z) \) or \( f \in CCH_n(\alpha) \).

**Remark 6.** From the above theorem we obtain \( CCH_n(\alpha) \subset CCH_0(\alpha) = CCH(\alpha) \) for all \( n \in \mathbb{N} \).
References


