A set of tangential approximation by meromorphic functions

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

The aim of this paper is to establish a closed subset $E$ of the complex plane $\mathbb{C}$, the interior $E^0$ of which forms one unbounded Gleason part, nevertheless $E$ is a set of tangential approximation by functions meromorphic in $\mathbb{C}$.

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1 Introduction

To state our main result we need some notations and facts. For arbitrary $A \subset \mathbb{C}$ we denote by $A^0, \partial A, \overline{A}$ and $A^c$ the interior, boundary, closure and complement of $A$ in $\mathbb{C}$, respectively. For a closed subset $E \subset \mathbb{C}$ let $\mathcal{A}(E)$ be the space of all complex valued functions which are continuous on $E$ and
holomorphic in the interior $E^0$ of $E$. For a compact $K \subset \mathbb{C}$ we denote by $\mathcal{R}(K)$ the set of all functions on $K$ which are uniform limits of functions rational in $\mathbb{C}$ without poles on $K$. Further, let $E$ be a relatively closed subset of a domain $D \subseteq \mathbb{C}$. Then the space $\mathcal{M}(E)$ denote the set of all functions on $E$ which are uniform limits of functions meromorphic in $D$ without poles on $E$.

**Theorem 1.** (Nersessian [5]) \( \mathcal{A}(E) = \mathcal{M}(E) \) if and only if $\mathcal{R}(E \cap K) = \mathcal{A}(E \cap K)$ for any closed disk $K \subset D$.

On the other hand we can base on the following sufficient conditions for the equality $\mathcal{A}(K) = \mathcal{R}(K)$ for any compact $K \subset \mathbb{C}$.

**Theorem 2.** (Mergelyan [4]) If $K^c$ has a finite number of components then $\mathcal{A}(K) = \mathcal{R}(K)$.

**Theorem 3.** (Vitushkin [6])) If the interior boundary of $K$ lies on countable many $C^2$ curves then $\mathcal{A}(K) = \mathcal{R}(K)$.

\( x \in \partial K \) is said to be an interior boundary point, if $x \notin \partial \Omega$ for any component $\Omega$ of $K^c$.

Theorem 2 is a consequence of Theorem 3, when the interior boundary of $K$ is empty.

**Definition 1.** A closed subset $E \subset \mathbb{C}$ is said to be a set of tangential (Carleman) approximation with functions meromorphic in $\mathbb{C}$, if for arbitrary functions $f$ and $\varepsilon$, where $f \in \mathcal{A}(E)$ and $\varepsilon \in C(E), \varepsilon > 0$, there exists a meromorphic function $g$ in $\mathbb{C}$ without poles on $E$ such that

\[ |f(z) - g(z)| < \varepsilon(z) \quad \text{for} \quad z \in E. \]
Definition 2.

(i) For a compact $K \subset \mathbb{C}$ we say that $x, y \in K$ are equivalent, $x \sim y$, if there exists a $c > 0$ such that
$$
\frac{1}{c} < \frac{u(x)}{u(y)} < c
$$
for any $u \in \text{Re}(R(K)), u > 0$.

(ii) Any equivalence class of $K$ is said to be a Gleason part of $R(K)$.

(iii) For a closed $E \subset \mathbb{C}$ a subset $G \subset E$ is called to be a Gleason part of $\mathcal{M}(E)$, if $K \cap G$ is a Gleason part of $R(K \cap E)$ for any closed disk $K$.

In the paper [1] the following condition is given for sets to be sets of tangential approximation.

Theorem 4. (Boivin [1]) Let $E \subset \mathbb{C}$ be closed. If for any closed disk $K$

(i) there exists a disk $\tilde{K} \supset K$ such that any Gleason part of $\mathcal{M}(E)$ that has a non empty intersection with $K$ lie in $\tilde{K}$,

(ii) if $A(K \cap E) = R(K \cap E)$,

then $E$ is a set of tangential approximation with functions meromorphic in $\mathbb{C}$.

In this paper we show that there exists a set $E$, the interior of which forms one unbounded Gleason part of $\mathcal{M}(E)$ (the condition (i) of Theorem 4 is not satisfied), but $E$ is a set of tangential approximation by functions meromorphic in $\mathbb{C}$. 
2 (L)-type sets

Let us set

\[ D(a,r) := \{ z \in \mathbb{C} : |z - a| < r \} \]
\[ D := D(0,1) \]
\[ C := \partial D. \]

**Definition 3.** A closed domain \( \mathcal{L} = \mathcal{L}(\{z_i\}_{i=1}^{\infty}, \{r_i\}_{i=1}^{\infty}) := \overline{\mathbb{D}} \setminus \bigcup_{i=1}^{\infty} D(z_i, r_i) \) is said to be an (L)-type set, if the sequences \( \{z_i\}_{i=1}^{\infty} \) and \( \{r_i\}_{i=1}^{\infty} \) satisfy the following conditions:

\( i \) \( |z_i| < 1, r_i < 1 - |z_i|, i = 1, 2, \ldots, \)

\( ii \) \( \{z_i\}_{i=1}^{\infty}' = C, \)

\( iii \) \( r_i + r_j < |z_i - z_j| \) for \( i \neq j. \)

(In (ii) "'" means the set of all cluster points).

**Definition 4.** An (L)-type set \( \mathcal{L} \) is called a uniqueness set if \( f \in \mathcal{A}(\mathcal{L}) \) and \( f(z) = 0 \) on \( C \) imply \( f(z) \equiv 0 \) on \( \mathcal{L}. \)

In [2] A.A. Gonchar has shown that there are (L)-type non-uniqueness sets. More precisely, the following proposition is true.

**Proposition 1.** For every \( \beta > 2 \) and \( \varepsilon > 0 \) there are

\( i \) \( (L)\)-type set \( \mathcal{L} \) with the property

\[ \sum_{i=1}^{\infty} \left( \frac{1}{\ln \frac{1}{r_i}} \right)^{\beta} < \varepsilon, \]
(ii) a function $\mu$ of the form
\begin{equation}
\mu(z) = \sum_{i=0}^{\infty} \frac{A_i}{z - z_i}
\end{equation}
(the serie converges uniform on $L$)
such that $\mu(z) = 0$ on $C$ and $\mu(z) \not\equiv 0$ on $L$.

**Corollary 1.** For arbitrary $\alpha > 0$ there exists an $(L)$ type set with a function $\mu$ satisfying the condition (ii) of Proposition 1 so that
\[ \sum_{i=1}^{\infty} r_i^\alpha < \infty. \]

In fact, for any $\alpha > 0, \beta > 2$ we have
\[ r_i^\alpha \left( \ln \frac{1}{r_i} \right)^\beta \to 0 \quad \text{when} \quad r_i \to 0. \]
Hence, $\sum_{i=1}^{\infty} \frac{1}{(\ln r_i)^\beta} < \varepsilon$ implies $\sum_{i=1}^{\infty} r_i^\alpha < \infty$.

**Remark 1.** The function $\mu$ is meromorphic in the unit disk.

In fact, on the circle $C(z_i, r_i), i = 0, 1, 2, \ldots$, the series $\sum_{k=0}^{\infty} \frac{A_k}{z - z_k}$ converges uniformly, and from the maximum principle follows that the series $\sum_{k=0}^{i-1} \frac{A_k}{z - z_k} + \sum_{k=i+1}^{\infty} \frac{A_k}{z - z_k}$ uniformly converges in the circle $D(z_i, r_i)$. Hence, $\mu$ is analytic in $D(z_i, r_i)$ except at the point $z = z_i$, where $\mu$ has a simple pole. Resuming, $\mu$ is meromorphic in unit circle with the simple poles $\{z_i\}_{i=0}^{\infty}$.

Below $L$ denotes an $(L)$-type non-uniqueness set with the property $\sum_{i=1}^{\infty} r_i < \infty$, and $\mu$ is the function from Proposition 1.

Let us set
\[ C_1 := \{ z = x + iy : |z| = 1, -1 \leq x < 0 \}; \]
\[ C_2 := \{ z = x + iy : |z| = 1, 0 < x \leq 1 \}. \]
Lemma 1. For any $L$ set there exists a meromorphic function $\nu(z)$ in the unit disk such that
\[
\lim_{L \ni z \to l} \nu(z) = \begin{cases} 
0, & l \in C_1 \\
1, & l \in C_2
\end{cases}.
\]

Proof. Let us set
\[
A_1 := L \setminus (D(1, \sqrt{2}) \cup C), \quad A_2 := L \setminus (D(-1, \sqrt{2}) \cup C), \quad F := A_1 \cup A_2,
\]
and take
\[
f(z) = \begin{cases} 
0, & z \in A_1 \\
1, & z \in A_2
\end{cases}.
\]
We have $f(z) \in \mathcal{A}(F)$. The complement of the intersection $F \cap K$ for any closed disk $K \subset \mathbb{D}$ consists of a finite number of components (since \(\{z_i\}' = C\)); hence, $F \cap K$ is a set of uniform approximation with rational functions (Theorem 2) which implies that $F$ is a set of uniform approximation with functions meromorphic in $\mathbb{D}$ (Theorem 1). Let us take the function $f/\mu$. The zeros of $\mu$ that lie in $A_2$ are denoted by $\xi_1, \ldots, \xi_n, \ldots$. Applying Mittag-Leffler’s theorem there is a meromorphic function $h(z)$ with poles at $\xi_1, \ldots, \xi_n, \ldots$ (and only this points), and with principal parts of the Laurent expansions coinciding with the corresponding principal parts of the function $1/\mu$. In that case we have $(f/\mu - h) \in \mathcal{A}(F)$. In $\mathbb{D}$ there exists a meromorphic function $v$ without poles on $F$ such that
\[
\left| \frac{f}{\mu} - h - v \right| < 1 \quad \text{on} \quad F,
\]
which gives us $|f - \mu(h+v)| < |\mu|$ on $F$. Because of $|\mu| \to 0$ when $z \to l \in C$, the in $\mathbb{D}$ meromorphic function $\nu := \mu(h+v)$ satisfies the assumption of the lemma.
Remark 2. For an $\mathcal{L}$ set for every point $z_0 \in \mathbb{C}$ the condition

\begin{equation}
\lim_{\delta \to 0} \frac{\sum_{D(z_i, r_i) \subset D(z_0, \delta)} r_i}{\delta} = 0
\end{equation}

is satisfied (cf. [2]); hence according to a result from [2] we get that the interior $X^0$ of $X := \overline{D(0,2)} \setminus \bigcup_{i=0}^{\infty} D(z_i, r_i)$ forms a Gleason part of $R(X)$.

3 The main result

Theorem 5. There exists a closed subset $E \subset \mathbb{C}$ so that $E^0$ forms an unbounded Gleason part of $\mathcal{M}(E)$ and $E$ is a set of tangential approximation by functions meromorphic in $\mathbb{C}$.

Proof. Consider the strip

$$\Pi := \{z = x + iy : -1 \leq y \leq 1\}$$

and the $\mathcal{L}$ set

$$\overline{D} \setminus \bigcup_{i=1}^{\infty} D(z_i, r_i)$$

and set

\begin{align*}
E & := (\Pi \setminus \bigcup_{n=-\infty}^{\infty} \bigcup_{i=1}^{\infty} D(z_i + 3n, r_i)) \setminus \bigcup_{n=-\infty}^{\infty} D(3n \pm i, \frac{1}{4}), \\
D'_n & := \overline{D}(0, 3n) \setminus (D(3m, 1) \cup D(3m \pm i, \frac{1}{4})), m = \pm n, n = 1, \ldots, \\
D''_n & := (D(0, 3n) \cup \overline{D}(3m, 1)) \setminus D(3m \pm i, \frac{1}{4}), m = \pm n, n = 1, \ldots.
\end{align*}

Because of Remark 2 the interior $E^0$ forms one Gleason part of $\mathcal{M}(E)$.

Let $f \in \mathcal{A}(E)$ be arbitrary, and $\varepsilon \in C(E), \varepsilon > 0$, tends to 0 if $|z| \to \infty$.

There exists a rational function $R_2(z)$ (Theorem 3) so that

$$|f(z) - R_2(z)| < \frac{\varepsilon(7)}{4(c + 1)}, z \in D''_2 \cap E,$$
where $c = ||\nu||_{\mathcal{L}}$ ($\nu$ is the function from Lemma 1).

Choose a rational function $Q_3(z)$ so that

$$|f(z) - R_2(z) - Q_3(z)| < \frac{\varepsilon(10)}{4(c + 1)}, z \in D' \cap E.$$ 

Set

$$\mu_3(z) := \begin{cases} 0, & z \in D'_2, \\ \nu(z), & z \in \overline{D(6,1)} \setminus D(6 \pm i, \frac{1}{4}), \\ 1 - \nu(z), & z \in \overline{D(-6,1)} \setminus D(-6 \pm i, \frac{1}{4}), \\ 1, & z \in (D'_3 \setminus (D'_2)^c) \cap E. \end{cases}$$

Clearly $\mu_3 \in \mathcal{A}(D'_2 \cup (D'_3 \cap E))$. According to Theorem 3 for any given $\delta > 0$ there exists a rational function $\tilde{\rho}_3$ so that

i) $|\tilde{\rho}_3|_{D'_2} < \delta$, 

ii) $|\tilde{\rho}_3 - 1|_{(D'_3 \setminus (D'_2)^c) \cap E} < \delta$,

iii) $|\tilde{\rho}_3|_{(D'_3 \cap E) \cup D'_2} < c + 1$.

Let $z_1^{(3)}, \ldots, z_{m_3}^{(3)}$ be the poles of $Q_3$ in $D'_1$ with multiplicities $\alpha_1^{(3)}, \ldots, \alpha_{m_3}^{(3)}$, respectively. According to the Cauchy integral formula for derivatives and (4) i), from the arbitrariness of $\delta$ we can assume that

(5) $|\tilde{\rho}_3^{(s)}(z_j^{(3)})| < \delta'$,

for arbitrary $\delta' > 0, s = 0, \ldots, \alpha_j^{(3)} - 1, j = 1, \ldots, m_3$. It is well-known that there exists a unique polynom $p_3$ of order $\sum_j^{m_3} \alpha_j^{(3)} - 1$, satisfying the conditions

$$p_3^{(s)}(z_j^{(3)}) = \tilde{\rho}_3^{(s)}(z_j^{(3)}), j = 1, \ldots, m_3, s = 0, \ldots, \alpha_j^{(3)} - 1.$$ 

In this connection the polynom has the form

$$p_3(z) = \sum_{j=1}^{m_3} \frac{\omega(z)}{(z - z_j^{(3)})^{\alpha_j^{(3)} - 1}} \sum_{s=0}^{\alpha_j^{(3)} - 1} A_{j,s}(z - z_j^{(3)})^s,$$
where

\[ \omega(z) = \prod_{j=1}^{m_j} (z - z_j^{(3)})^{\alpha_j^{(3)}}, \]

\[ A_{j,s} = \sum_{\nu=0}^{s} \frac{1}{\nu!(s-\nu)!} \hat{\rho}_3^{(\nu)}(z_j^{(3)}) \left[ \frac{d^{s-\nu}}{dz^{s-\nu}} \frac{(z - z_j^{(3)})^{\alpha_j^{(3)}}}{\omega(z)} \right]_{z=z_j}. \]

From (5) it follows that \(|A_{j,s}| \) and hence also \(|p_3|_K\) for any compact \(K \subset \mathbb{C}\) can be assumed arbitrarily small. Summing up, it can be assumed that the rational function \(\hat{\rho}_3 - \rho_3\) satisfies the conditions

\[ \rho_3^{(s)}(z_j^{(3)}) = 0, s = 0, 1, \ldots, \alpha_j^{(3)} - 1, j = 1, \ldots, m_j, \]

\[ |\rho_3|_{D'_2} < \varepsilon, \]

\[ |\rho_3 - 1|_{(D'_2 \setminus D''_1) \cap E} < \varepsilon, \]

\[ |\rho_3|_{(D''_1 \cap E) \cup D'_2} < c + 1 \]

for any \(\varepsilon > 0\). Observe that the function \(\rho_3\) is taken so that the rational function \(R_3 = \rho_3Q_3\) has no poles in \(D'_1\). Taking \(\varepsilon\) in (6) sufficiently small, we can assume that the rational function \(R_3\) satisfies the conditions

\[ |R_3| < \frac{1}{2^4}, z \in D'_1, \]

\[ |f(z) - R_2(z) - R_3(z)| < \varepsilon(4), z \in D''_1 \cap E, \]

\[ |f(z) - R_2(z) - R_3(z)| < \frac{\varepsilon(10)}{4(c+1)}, z \in (D''_3 \setminus D''_2) \cap E, \]

\[ |f(z) - R_2(z) - R_3(z)| \leq |f(z) - R_2(z)| + |R_3(z)| \]

\[ < \frac{\varepsilon(7)}{4(c+1)} + (c+1) |Q_3(z)| \]

\[ < \frac{\varepsilon(7)}{4(c+1)} + (c+1) \left( \frac{\varepsilon(7)}{4(c+1)} + \frac{\varepsilon(10)}{4(c+1)} \right), \]

\[ < \varepsilon(7), z \in (D''_2 \setminus D''_1) \cap E. \]
Let now for any \( n > 3 \) the functions \( R_2, \ldots, R_n \) are taken so that

\[
\begin{align*}
    i) & \quad |R_k(z)| < \frac{1}{2^k}, z \in D_{k-2}', k = 3, \ldots, n, \\
    (7) ii) & \quad |f(z) - R_2(z) - \ldots - R_n(z)| < \varepsilon(3k + 1), z \in (D_k'' \setminus D_{k-1}'') \cap E, \quad k = 1, \ldots, n - 1, D_0'' = \emptyset, \\
    iii) & \quad |f(z) - R_2(z) - \ldots - R_n(z)| < \frac{\varepsilon(3n + 1)}{4(c + 1)}, z \in (D_n'' \setminus D_{n-1}'') \cap E.
\end{align*}
\]

According to Theorem 3 there exists a rational function \( Q_{n+1} \) satisfying the condition

\[
(8) \quad |f(z) - R_2(z) - \ldots - R_n(z) - Q_{n+1}(z)| < \frac{\varepsilon(3n + 1) + 1}{4(c + 1)}, z \in D_{n+1}'' \cap E.
\]

Arguing as for the construction of the function \( \rho_3 \) we get a rational function \( \rho_{n+1} \) satisfying the conditions

\[
(9) \quad |\rho_{n+1}|_{D_n''} < \varepsilon, \\
|\rho_{n+1} - 1|_{(D_n'' \setminus (D_0'' \cap E)) \cap E} < \varepsilon, \\
|\rho_{n+1}|_{(D_n'' \setminus E) \cup D_n''} < c + 1,
\]

where \( \varepsilon \) can be arbitrarily small. In particular, we can assume \( \varepsilon \) so small that the rational function \( R_{n+1} = \rho_{n+1}Q_{n+1} \) satisfies the conditions

\[
(10) \quad i) \quad |R_{n+1}(z)| < \frac{1}{2^{n+1}}, z \in D_{n-1}'.
\]

\[
ii) \quad |f(z) - R_2(z) - \ldots - R_{n+1}(z)| < \varepsilon(3k + 1), z \in (D_k'' \setminus D_{k-1}'') \cap E, \quad k = 1, \ldots, n - 1, D_0'' = \emptyset, \\
iii) \quad |f(z) - R_2(z) - \ldots - R_{n+1}(z)| < \frac{\varepsilon(3n + 1) + 1}{4(c + 1)}, z \in (D_{n+1}'' \setminus D_n'') \cap E.
\]
According to the relations (7) iii), (8) and (9), we have

\[ |f(z) - R_2(z) - \ldots - R_{n+1}(z)| \leq |f(z) - \ldots - R_n(z)| + |R_{n+1}(z)| < \varepsilon(3n + 1) \]

\[ + (c+1) \left( \frac{\varepsilon(3(n+1)+1)}{4(c+1)} + \frac{\varepsilon(3n+1)}{4(c+1)} \right) < \varepsilon(3n + 1). \]

¿From (10) and (11) it follows that the relations (7) are true if \( n \) is replaced by \( n + 1 \). By induction, we can assume that there exists a sequence \( \{R_n\}_{n=2}^\infty \) of rational functions satisfying the conditions

\[ |R_k(z)| < \frac{1}{2^k}, z \in D_{k-2}^\prime, k = 3, 4, \ldots. \]

Thus it follows that the serie \( G = \sum_{n=2}^\infty R_n \) uniformly converges on any compact subset of \( \mathbb{C} \) after dropping a finite number of summands. Since all summands are rational functions, \( G \) is meromorphic in \( \mathbb{C} \). On the other hand, since for any number \( k = 1, 2, \ldots \) the relation (7) ii) is valid for all numbers \( n, n > k \), passing to the limit when \( n \to \infty \), for any \( z \in E \) we get

\[ |f(z) - G(z)| < \varepsilon(z). \]

The theorem is proved.

References


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