Estimations of the Error for Two-point Formula via Pre-Grüss Inequality

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Generalization of estimation of two-point formula is given, by using pre-Grüss inequality.

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1 Introduction

In the recent paper [4] N. Ujević use the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson’s quadrature rule. In fact, he proved the next as his main result:
Theorem 1. If \( g, h, \Psi \in L_2(0,1) \) and \( \int_0^1 \Psi(t)dt = 0 \) then we have

\[
|S_{\Psi}(g, h)| \leq S_{\Psi}(g, g)^{1/2}S_{\Psi}(h, h)^{1/2},
\]

where

\[
S_{\Psi}(g, h) = \int_0^1 g(t)h(t)dt - \int_0^1 g(t)dt \int_0^1 h(t)dt - \int_0^1 g(t)\Psi_0(t)dt \int_0^1 h(t)\Psi_0(t)dt
\]

and \( \Psi_0(t) = \Psi(t)/\|\Psi\|_2 \).

Further, he gave some improvements of the Simpson’s inequality. For example he get:

Theorem 2. Let \( I \subset \mathbb{R} \) be a closed interval and \( a, b \in \text{Int}I, \ a < b \). If \( f : I \to \mathbb{R} \) is an absolutely continuous function with \( f' \in L_2(a,b) \) then we have

\[
\left| \frac{b-a}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^{3/2}}{6}K_1,
\]

where

\[
K_1^2 = \|f''\|_2^2 - \frac{1}{b-a}\left( \int_a^b f'(t)dt \right)^2 - \left( \int_a^b f'(t)\Psi_0(t)dt \right)^2
\]

and \( \Psi(t) = t - \frac{a+b}{2}, \Psi_0(t) = \Psi(t)/\|\Psi\|_2 \).

In this paper using the Theorem 1 we will give the similar result for Euler two-point formula and for functions whose derivative of order \( n, \ n \geq 1 \), is from \( L_2(0,1) \) space. We will use interval \([0,1]\) because of simplicity and since it involves no loss in generality.
2 Estimations of the error for Euler two-point formula

In the recent paper [3] the following identity, named Euler two-point formula, has been proved. For \( n \geq 1, \ x \in [0, \frac{1}{2}] \) and every \( t \in [0,1] \) we have

\[
\int_0^1 f(t)dt = D(x) - T_n(x) + R_n(x)
\]

where

\[
D(x) = \frac{1}{2} [f(x) + f(1 - x)],
\]

\( T_0(x) = 0 \) and

\[
T_m(x) = \frac{1}{2} \sum_{k=1}^{m} \frac{\hat{B}_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right],
\]

for \( 1 \leq m \leq n \) and \( x \in [0, \frac{1}{2}] \), while

\[
\hat{B}_k(x) = B_k(x) + B_k(1 - x), \ k \geq 1,
\]

\[
R_n(x) = \frac{1}{2(n!)} \int_0^1 G_n^x(t) f^{(n)}(t)dt
\]

and

\[
G_n^x(t) = B_n^x(x - t) + B_n^x(1 - x - t), \ t \in \mathbb{R}.
\]

The identity holds for every function \( f : [0, 1] \to \mathbb{R} \) such that \( f^{(n-1)} \) is a continuous function of bounded variation on \([0,1]\). The functions \( B_k(t) \) are the Bernoulli polynomials, \( B_k = B_k(0) \) are the Bernoulli numbers, and \( B_k^*(t), \ k \geq 0, \) are periodic functions of period 1, related to the Bernoulli polynomials as

\[
B_k^*(t) = B_k(t), \ 0 \leq t < 1 \quad \text{and} \quad B_k^*(t + 1) = B_k^*(t), \ t \in \mathbb{R}.
\]
The Bernoulli polynomials $B_k(t), k \geq 0$ are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t), \ k \geq 1; \ B_0(t) = 1, \ B_k(t+1) - B_k(t) = kt^{k-1}, \ k \geq 0.$$ 

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of $-1$ at each integer. It follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ are continuous functions for $k \geq 2$. We get

$$B_k^*(t) = kB_{k-1}^*(t), \ k \geq 1$$

for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

**Theorem 3.** If $f : [0, 1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0, 1)$ then we have

$$\left| \int_0^1 f(t)dt - D(x) + T_n(x) \right| \leq \frac{1}{2} \left[ \frac{2(-1)^{n-1}}{(2n)!} [B_{2n} + B_{2n}(1 - 2x)] \right]^{1/2} K,$$

where

$$K^2 = \| f^{(n)} \|_2^2 - \left( \int_0^1 f^{(n)}(t)dt \right)^2 - \left( \int_0^1 f^{(n)}(t)\Psi_0(t)dt \right)^2.$$

For $n$ even

$$\Psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}] \\ -1, & t \in \left(\frac{1}{2}, 1\right) \end{cases},$$

while for $n$ odd we have

$$\Psi(t) = \begin{cases} t + \frac{B_{n+1}(\frac{1}{2} + x)}{2(B_{n+1}(x) - B_{n+1}(\frac{1}{2} + x))}, & t \in [0, \frac{1}{2}], \\ t + \frac{B_{n+1}(\frac{1}{2} + x) - 2B_{n+1}(x)}{2(B_{n+1}(x) - B_{n+1}(\frac{1}{2} + x))}, & t \in \left(\frac{1}{2}, 1\right). \end{cases}$$
Proof. It is not difficult to verify that

\( \int_0^1 G_n(t) dt = 0, \) (9)

\( \int_0^1 \Psi(t) dt = 0, \) (10)

\( \int_0^1 G_n(t) \Psi(t) dt = 0. \) (11)

From (4), (9) and (11) it follows that

\( \int_0^1 f(t) dt - D(x) + T_n(x) = \frac{1}{2(n!)} \int_0^1 G_n^x(t) f^{(n)}(t) dt - \frac{1}{2(n!)} \int_0^1 G_n^x(t) dt \int_0^1 f^{(n)}(t) dt - \frac{1}{2(n!)} \int_0^1 G_n^x(t) \Psi_0(t) dt \int_0^1 f^{(n)}(t) \Psi_0(t) dt = \frac{1}{2(n!)} S_\Psi(G_n^x, f^{(n)}). \) (12)

Using (12) and (1) we get

\[ \left| \int_0^1 f(t) dt - D(x) + T_n(x) \right| \leq \frac{1}{2(n!)} S_\Psi(G_n^x, G_n^x)^{1/2} S_\Psi(f^{(n)}, f^{(n)})^{1/2}. \] (13)

We also have (see [3])

\[ S_\Psi(G_n^x, G_n^x) = \|G_n^x\|^2 - \left( \int_0^1 G_n^x(t) dt \right)^2 - \left( \int_0^1 G_n^x(t) \Psi_0(t) dt \right)^2 = \]

\[ (-1)^{n-1} \frac{2(n!)^2}{(2n)!} [B_{2n} + B_{2n}(1 - 2x)] \] (14)

and

\[ S_\Psi(f^{(n)}, f^{(n)}) = \|f^{(n)}\|^2 - \left( \int_0^1 f^{(n)}(t) dt \right)^2 - \left( \int_0^1 f^{(n)}(t) \Psi_0(t) dt \right)^2 = K^2. \] (15)

From (13)-(15) we easily get (7).
Remark 1. Function $\Psi(t)$ can be any function which satisfies conditions
\[ \int_0^1 \Psi(t) dt = 0 \quad \text{and} \quad \int_0^1 G_n^x(t) \Psi(t) dt = 0. \]
Because $G_n^x(1 - t) = (-1)^n G_n^x(t)$ (see [3]), for $n$ even we can take function $\Psi(t)$ such that $\Psi(1 - t) = -\Psi(t)$.

For $n$ odd we have to calculate $\Psi(t)$ and with not lost in generality in our theorem we take the form
\[ \Psi(t) = \begin{cases} t + a, & t \in [0, \frac{1}{2}] , \\ t + b, & t \in (\frac{1}{2}, 1] . \end{cases} \]

Remark 2. For $n = 1$ in Theorem 3 we have
\[ \left| \int_0^1 f(t) dt - D(x) \right| \leq \frac{1}{2} \left[ \frac{1}{3} - 2x + 4x^2 \right]^{1/2} K, \]
while
\[ \Psi(t) = \begin{cases} t + \frac{1-12x^2}{24x-6}, & t \in [0, \frac{1}{2}] , \\ t + \frac{12x^2-24x+5}{24x-6}, & t \in (\frac{1}{2}, 1] . \end{cases} \]

Also, for $n = 2$ we have
\[ \left| \int_0^1 f(t) dt - D(x) \right| \leq \frac{1}{2} \left[ \frac{1}{180} - \frac{x^2}{3} + \frac{4x^3}{3} - \frac{4x^4}{3} \right]^{1/2} K, \]
while
\[ \Psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}] , \\ -1, & t \in (\frac{1}{2}, 1] . \end{cases} \]

If in Theorem 3 we choose $x = 0, 1/2, 1/3, 1/4$ we get inequality related to the trapezoid, the midpoint, the two-point Newton-Cotes and the two-point MacLaurin formula:

Corollary 1. If $f : [0, 1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0, 1)$ then we have
\[ \left| \int_0^1 f(t) dt - \frac{1}{2} [f(0) + f(1)] + T_n(0) \right| \leq \left[ \frac{(-1)^{n-1}}{(2n)!} B_{2n} \right]^{1/2} K, \]
where \( T_0(0) = 0 \),

\[
T_n(0) = \sum_{k=1}^{[n/2]} \frac{B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]
\]

and

\[
K^2 = \| f^{(n)} \|^2_2 - \left( \int_0^1 f^{(n)}(t)dt \right)^2 - \left( \int_0^1 f^{(n)}(t)\Psi_0(t)dt \right)^2 .
\]

For \( n \) even

\[
\Psi(t) = \begin{cases} 
1, & t \in [0, \frac{1}{2}] , \\
-1, & t \in (\frac{1}{2}, 1] ,
\end{cases}
\]

while for \( n \) odd we have

\[
\Psi(t) = \begin{cases} 
t + \frac{2^{-n-1}}{2^{2^{-n-1}-n}}, & t \in [0, \frac{1}{2}] , \\
t + \frac{2^{-n-3}}{2^{2^{-n-1}-n}}, & t \in (\frac{1}{2}, 1] .
\end{cases}
\]

Remark 3. For \( n = 1 \) in Corollary 1 we have

\[
\left| \int_0^1 f(t)dt - \frac{1}{2} [f(0) + f(1)] \right| \leq \frac{K}{2\sqrt{3}},
\]

while

\[
\Psi(t) = \begin{cases} 
t - \frac{1}{6}, & t \in [0, \frac{1}{2}] , \\
t - \frac{5}{6}, & t \in (\frac{1}{2}, 1] .
\end{cases}
\]

Corollary 2. If \( f : [0, 1] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is absolutely continuous function with \( f^{(n)} \in L_2(0,1) \) then we have

\[
\left| \int_0^1 f(t)dt - f \left( \frac{1}{2} \right) + T_n \left( \frac{1}{2} \right) \right| \leq \left[ \frac{(-1)^{n-1}}{(2n)!} B_{2n} \right]^{1/2} K,
\]

where \( T_0 \left( \frac{1}{2} \right) = 0 \),

\[
T_n \left( \frac{1}{2} \right) = \sum_{k=1}^{[n/2]} \frac{(2^{-2k} - 1)B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]
\]

\[
= \sum_{k=1}^{[n/2]} \frac{(2^{-2k} - 1)B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]
\]
and
\[ K^2 = \|f^{(n)}\|_2^2 - \left( \int_0^1 f^{(n)}(t) dt \right)^2 - \left( \int_0^1 f^{(n)}(t) \Psi_0(t) dt \right)^2. \]

For \( n \) even
\[ \Psi(t) = \begin{cases} 
1, & t \in [0, \frac{1}{2}], \\
-1, & t \in (\frac{1}{2}, 1], 
\end{cases} \]
while for \( n \) odd we have
\[ \Psi(t) = \begin{cases} 
t + \frac{1}{2^{1-n-4}}, & t \in [0, \frac{1}{2}], \\
t + \frac{3^{1-n} - 2^{1-n}}{2^{1-n-4}}, & t \in (\frac{1}{2}, 1]. 
\end{cases} \]

Remark 4. For \( n = 1 \) in Corollary 2 we have
\[ \left| \int_0^1 f(t) dt - f\left(\frac{1}{2}\right) \right| \leq \frac{K}{2\sqrt{3}}, \]
while
\[ \Psi(t) = \begin{cases} 
t - \frac{1}{3}, & t \in [0, \frac{1}{2}], \\
t - \frac{2}{3}, & t \in (\frac{1}{2}, 1]. 
\end{cases} \]

Corollary 3. If \( f : [0, 1] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is absolutely continuous function with \( f^{(n)} \in L_2(0, 1) \) then we have
\[ \left| \int_0^1 f(t) dt - \frac{1}{2} \left[ f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right] + T_n\left(\frac{1}{3}\right) \right| \leq \]
\[ \leq \frac{1}{2} \left[ \frac{(-1)^{n-1}}{(2n)!} (1 + 3^{1-2n}) B_{2n} \right]^{1/2} K, \]
where \( T_0\left(\frac{1}{3}\right) = 0, \)
\[ T_n\left(\frac{1}{3}\right) = \frac{1}{2} \sum_{k=1}^{\lceil n/2 \rceil} \frac{(3^{1-2k} - 1) B_{2k}}{(2k)!} [f^{(2k)}(1) - f^{(2k-1)}(0)]. \]
and
\[ K^2 = \| f^{(n)} \|_2^2 - \left( \int_0^1 f^{(n)}(t) dt \right)^2 - \left( \int_0^1 f^{(n)}(t) \Psi_0(t) dt \right)^2. \]

For \( n \) even
\[ \Psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}] , \\ -1, & t \in \left( \frac{1}{2}, 1 \right] , \end{cases} \]

while for \( n \) odd we have
\[ \Psi(t) = \begin{cases} t + \frac{1-2^n}{2^{n+1}-2}, & t \in [0, \frac{1}{2}], \\ t + \frac{1-3-2^n}{2^{n+1}-2}, & t \in \left( \frac{1}{2}, 1 \right]. \]

**Remark 5.** For \( n = 1 \) in Corollary 3 we have
\[ \left| \int_0^1 f(t) dt - \frac{1}{2} \left[ f \left( \frac{1}{3} \right) + f \left( \frac{2}{3} \right) \right] \right| \leq \frac{K}{6}, \]

while
\[ \Psi(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}], \\ t - \frac{5}{6}, & t \in \left( \frac{1}{2}, 1 \right]. \]

**Corollary 4.** If \( f : [0, 1] \rightarrow \mathbb{R} \) is such that \( f^{(2m-1)} \) is absolutely continuous function with \( f^{(2m)} \in L_2(0,1) \) then we have
\[ \left| \int_0^1 f(t) dt - \frac{1}{2} \left[ f \left( \frac{1}{4} \right) + f \left( \frac{3}{4} \right) \right] + T_{2m} \left( \frac{1}{4} \right) \right| \leq \left[ \frac{-2^{-4m}}{(4m)!} B_{4m} \right]^{1/2} K, \]

where \( T_0 \left( \frac{1}{4} \right) = 0, \)
\[ T_{2m} \left( \frac{1}{4} \right) = \sum_{k=1}^{m} \frac{2^{-2k}(21^{2k-2k}-1)B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right] \]

and
\[ K^2 = \| f^{(2m)} \|_2^2 - \left( \int_0^1 f^{(2m)}(t) dt \right)^2 - \left( \int_0^1 f^{(2m)}(t) \Psi_0(t) dt \right)^2, \]
while

\[ \Psi(t) = \begin{cases} 
1, & t \in \left[0, \frac{1}{2}\right], \\
-1, & t \in \left(\frac{1}{2}, 1\right]. 
\end{cases} \]

References


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