Proof of the best bounds in Wallis’ inequality

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Let $n \geq 1$ be an integer, then

$$\frac{1}{\sqrt{\pi(n + 4\pi^{-1} - 1)}} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{\pi(n + 1/4)}}.$$ 

The constants $4\pi^{-1} - 1$ and $1/4$ are the best possible.

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The sine has the infinite product representation

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right).$$ (1)
Taking in (1) \( x = \pi/2 \) gives well known the Wallis formula

\[
\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.
\]

Motivated by (2), Kazarinoff [2] proved that

\[
\frac{1}{\sqrt{\pi(n + \frac{1}{2})}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{\pi(n + \frac{1}{4})}}
\]

for \( n \in \mathbb{N} \), the set of positive integers. We here show that, for \( n \in \mathbb{N} \),

\[
\frac{1}{\sqrt{\pi(n + 4\pi^{-1} - 1)}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{\pi(n + 1/4)}},
\]

improving the lower bound and confirming the upper in (3), by a very simple argument. We also prove that the bounds in (4) are the best possible.

**Proof.** It is clear that

\[
\Gamma(n + 1) = n!, \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}, \quad 2^n n! = (2n)!!. \]

To prove the right hand inequality of (4), it suffices to show that

\[
R_n = \frac{\Gamma\left(n + \frac{1}{2}\right) \sqrt{n + \frac{1}{4}}}{\Gamma(n + 1)} < 1.
\]

Using the recurrence relation for the gamma function \( \Gamma(x + 1) = x \Gamma(x) \) we conclude that

\[
\frac{R_n}{R_{n+1}} = \sqrt{\frac{n + \frac{1}{4}}{n + \frac{5}{4}}} \frac{n + 1}{n + \frac{1}{2}} < 1 \quad \text{for} \quad n \geq 1.
\]

Hence, the sequence \( \{R_n\}_{n=1}^{\infty} \) is strictly increasing with \( n \in \mathbb{N} \).
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From the asymptotic expansion [1, p. 257]

$$x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O(x^{-2}),$$

we conclude that \( \lim_{n \to \infty} R_n = 1 \), thus inequality (5) holds for all \( n \in \mathbb{N} \).

The left hand side of inequality (4) is equivalent to

$$L_n = \frac{\Gamma(n + \frac{1}{2}) \sqrt{n + \frac{4}{\pi} - 1}}{\Gamma(n + 1)} \geq 1.$$

It is easy to see that

$$\frac{L_n}{L_{n+1}} = \sqrt{\frac{n + \frac{3}{2} - 1}{n + \frac{4}{\pi}} \frac{n + 1}{n + \frac{1}{2}}} > 1 \quad \text{for} \quad n \geq 2.$$

Hence, the sequence \( \{L_n\}_{n=1}^\infty \) is strictly decreasing for \( n \geq 2 \). By (6), we conclude that \( \lim_{n \to \infty} L_n = 1 \), thus inequality (7) holds strictly for all \( n \geq 2 \). Clearly, the sign of equality in (7) holds for \( n = 1 \). The proof is complete.

References


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