First order nonlinear differential superordination

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Dedicated to Professor Emil C. Popa on his 60th anniversary

Abstract

In this paper we shall extend the first-order linear differential superordination defined by the authors in [2] to a first-order nonlinear differential superordination of the form:

\[ U \subset \{ \lambda(z)zp'(z) + \mu(z)p^2(z) + p(z); \; z \in U \}. \]

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1 Introduction

Let \( \Omega \) be any set in the complex plane \( \mathbb{C} \), let \( p \) be analytic in the unit disk \( U \) and let \( \psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C} \). In a series of articles the authors and
many others [1] have determined properties of functions \( p \) that satisfy the differential subordination

\[
\{ \psi(p(z), zp'(z), z^2p'(z); z) \mid z \in U \} \subset \Omega.
\]

In this article we consider the dual problem of determining properties of functions \( p \) that satisfy the differential superordination

\[
\Omega \subset \{ \psi(p(z), zp'(z), z^2p'(z); z) \mid z \in U \}.
\]

This problem was introduced in [2].

We let \( \mathcal{H}(U) \) denote the class of holomorphic functions in the unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For \( a \in \mathbb{C} \) and \( n \in \mathbb{N} \) we let

\[
\mathcal{H}[a, n] = \{ f \in \mathcal{H}(U), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U \}.
\]

For \( 0 < r < 1 \), we let \( U_r = \{ z \in \mathbb{C}, |z| < r \} \).

**Definition 1.** [2] Let \( \varphi : \mathbb{C}^2 \times U \to \mathbb{C} \) and let \( h \) be analytic in \( U \). If \( p \) and \( \varphi(p(z), zp'(z); z) \) are univalent in \( U \) and satisfy the (first-order) differential superordination

\[
h(z) \prec \varphi(p(z), zp'(z); z)
\]

then \( p \) is called a solution of the differential superordination. An analytic function \( q \) is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if \( q \prec p \) for all \( p \) satisfying (1). A univalent subordinant \( \tilde{q} \) that satisfies \( q \prec \tilde{q} \) for all subordinants \( q \) of (1) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of \( U \).
For $\Omega$ a set in $\mathbb{C}$, with $\varphi$ and $p$ as given in Definition 1, suppose (1) is replaced by

$$(1') \quad \Omega \subset \{ \varphi(p(z), zp'(z); z) | z \in U \}.$$  

Although this more general situation is a "differential containment", the condition in (1) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extended to this generalization.

**Definition 2.** [2] We denote by $Q$ the set of functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty \}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of $Q$ for which $f(0) = a$ is denoted by $Q(a)$.

**Definition 3.** [2] Let $Q$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^2 \times \overline{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$(2) \quad \varphi\left(q(z), \frac{zq'(z)}{m}; \zeta \right) \in \Omega$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$.

In order to prove the new results we shall use the following lemma:

**Lemma A.** [2] Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{H}[a, n]$, $\varphi : \mathbb{C}^2 \times \overline{U} \rightarrow \mathbb{C}$, and suppose that

$$(3) \quad \varphi(q(z), tzq'(z); \zeta) \in \Omega,$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \frac{1}{n} \leq 1$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z); z)$ is univalent in $U$, then

$$\Omega \subset \{ \varphi(p(z), zp'(z); z) | z \in U \} \text{ implies } q(z) \preceq p(z).$$
Lemma B. [1, Lemma 2.2.d p. 24] Let $q \in Q$, with $q(0) = a$, and let

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$$

be analytic in $U$ with $p(z) \not\equiv a$ and $n \geq 1$. If $p$ is not subordinate to $q$, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$, $r_0 < 1$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$, and

(iii) $\operatorname{Re} \left( \frac{z_0 p'(z_0)}{p'(z_0)} + 1 \right) \geq m \operatorname{Re} \left[ \frac{\zeta_0 q'(\zeta_0)}{q'(\zeta_0)} + 1 \right]$.

2 Main results

Theorem 1. Let $\lambda, \mu : \overline{U} \to \mathbb{C}$ with $|\lambda(\zeta)| \leq 1$, $|\mu(\zeta)| \leq 1$, $\zeta = e^{i\theta}$, $p \in \mathcal{H}[0,1] \cap Q$ and let

$$\lambda(z)p'(z) + \mu(z)p^2(z) + p(z)$$

be univalent in unit disk $U$.

If

$$U \subset \{ \lambda(z)p'(z) + \mu(z)p^2(z) + p(z); \ z \in U \}$$

or

$$z \prec \lambda(z)p'(z) + \mu(z)p^2(z) + p(z)$$

then

$$U_r \subset p(U) \text{ or } rz \prec p(z),$$
where \( r \) is given by
\[
(4) \quad r = \sqrt{2} - 1.
\]

**Proof.** Let \( \Omega = \{ w \in \mathbb{C} \mid |w| < 1\} = U \) and let \( q(z) = rz, q(U) = U_r = \{ w \in \mathbb{C} \mid |w| < r\} = \Delta. \)

We let
\[
\varphi(p(z), zp'(z); z \mid z \in U) = \lambda(z)zp'(z) + \mu(z)p^2(z) + p(z).
\]

In order to apply Lemma A to prove this result we only need to show that admissibility condition holds
\[
|\varphi(q(z), tzq'(z); \zeta)| = |\lambda(\zeta)tq'(z) + \mu(\zeta)q^2(z) + q(z)|
\]
\[
= |\lambda(\zeta)trz + \mu(\zeta)r^2z^2 + rz| = |z| \lambda(\zeta) + \mu(\zeta)r z + 1| + |z| |\lambda(\zeta)| + |\mu(\zeta)| = r^2 + 2r - 1.
\]

Since \( |\varphi(q(z), tzq'(z); \zeta)| \in U \), by using Lemma A it results \( U_r \subset p(U) \), or \( rz < p(z) \).

**Remark 1.** If \( \mu(z) \equiv 0 \), we obtain the result from [2, Theorem 10]. If \( \lambda(z) = -z, \mu(z) \equiv 1 \) then the differential equation
\[
-z \cdot zq'(z) + q^2(z) + q(z) = z
\]
has the univalent solution \( q(z) = z \). Hence from the sharp form of Theorem 1 we obtain the following result.

**Corollary 1.** If \( p \in \mathcal{H}[0, 1] \cap Q \) and \( -z^2 p'(z) + p^2(z) + p(z) \) is univalent, then \( z < -z^2 p'(z) + p^2(z) + p(z) \), implies
\[
z < p(z), \quad z \in U.
\]
The function $z$ is the best subordinant.

**Remark 2.** If $\lambda(z) \equiv 1$, $\mu(z) \equiv 0$, we obtain the result from [2, Corollary 10.1].

If $\lambda(z) = \frac{-z}{4}$, $\mu(z) = \frac{1}{4}$, then the differential equation

$$-rac{z}{4} z q'(z) + \frac{1}{4} q'^2(z) + q(z) = z$$

has the univalent solution $q(z) = z$. Hence from the sharp form of Theorem 1 we obtain the following result.

**Corollary 2.** If $p \in H[0, 1] \cap Q$ and

$$-rac{z}{4} z p'(z) + \frac{1}{4} p^2(z) + p(z)$$

is univalent, then

$$z < -\frac{z^2}{4} p'(z) + \frac{1}{4} p^2(z) + p(z)$$

implies

$$z < p(z), \quad z \in U.$$

The function $z$ is the best subordinant.

**Theorem 2.** Let $N > 1$, $M > 0$, $\lambda, \mu : \mathbb{U} \to \mathbb{C}$, with $|\lambda(z)|^2 + \text{Re} \lambda(z) \geq 0$, $p \in H[0, 1] \cap Q$, and

$$M[|\lambda(z) + 1| - M|\mu(z)|] \geq N.$$

If

$$\lambda(z) z p'(z) + \mu(z) p^2(z) + p(z) \prec Nz$$

then

$$p(z) \prec Mz.$$
Proof. If we let
\[ w(z) = \lambda(z)zp'(z) + \mu(z)p^2(z) + p(z), \]
then
\[ |w(z)| < N. \quad (5) \]

Let \( q(z) = Mz \). If \( p(z) \not< q(z) \), then by Lemma B there exist \( z_0 \in U \), \( \zeta \in \partial U \) and \( m > 1 \) such that \( p(z_0) = q(\zeta) = M\zeta \) and \( z_0p'(z_0) = \zeta q'(\zeta) = mM\zeta \).

For \( z_0 \in U \) we have
\[ E = |w(z_0)| = |\lambda(z_0)z_0p'(z_0) + \mu(z_0)p^2(z_0) + p(z_0)| = \]
\[ = |\lambda(z_0)mNM\zeta + \mu(z_0)M^2\zeta + M\zeta| = \]
\[ = M|\lambda(z_0)m + \mu(z_0)M\zeta + 1| \geq M[|\lambda(z_0)m + 1| - M|\mu(z_0)|]. \]

From \( |\lambda(z)|^2 + \text{Re} \lambda(z) \geq 0 \) we have
\[ |\lambda(z_0)m + 1| > |\lambda(z_0) + 1|. \]

Hence
\[ E \geq M[|\lambda(z_0) + 1| - M|\mu(z_0)|] \geq N. \]

Since this contradicts (5), we obtain the desired result \( p(z) \prec Mz \).

Under the conditions of Theorem 1 and Theorem 2 we have the following sandwich type result.

Corollary 3. If
\[ z \prec \lambda(z)zp'(z) + \mu(z)p^2(z) + p(z) \prec Nz \]
then

\[ rz < p(z) < Mz, \]

where \( r \) is given by (4) and \( M > 0 \).

References


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