Inequalities concerning starlike functions and their n-th root

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Abstract

If $A$ is the class of all analytic functions in the complex unit disc $\Delta$, of the form:

$$f(z) = z + a_2 z^2 + \cdots$$

and if $f \in A$ satisfies in $\Delta$ the condition:

$$\Re \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right|$$

then $\sqrt[n]{f(z)/z} \geq (n+1)/(n+2)$. We show also that if $f$ is starlike in $\Delta$ (i.e. $\Re zf'(z)/f(z) > 0$ in $\Delta$), then $\sqrt[n]{f(z)/z} > n/(n+2)$.

Mathematical Subject Classification: Primary: 30C45

Keywords: starlike function, uniformly starlike function, uniformly convex function.

1 Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and let $A$ be the set of all analytic functions in $\Delta$, having the power series development:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$
The subclass $\text{ST}$ of $A$ consists of functions $f$ which satisfy in $\Delta$ the condition:
\[ \Re \frac{zf'(z)}{f(z)} > 0 \quad z \in \Delta \]

$\text{ST}$ is named the class of **starlike functions** in $A$

The subclass $\text{CV}$ of $\text{ST}$ (named the class of **convex functions** in $A$) consists of functions $f \in A$ which satisfy in $\Delta$ the condition:
\[ \Re \left[ 1 + \frac{zf'(z)}{f'(z)} \right] > 0 \quad z \in \Delta \]

We denote also by $\text{QUST}$ (**quasi-uniformly starlike functions**) the subclass of $\text{ST}$ which contains the functions satisfying the condition:
\[ \Re \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f'(z)} - 1 \right| \]

and by $\text{UCV}$ (**uniformly convex functions**) the subclass of $\text{CV}$ which contains functions satisfying the condition:

(1) \[ \Re [1 + \frac{zf'(z)}{f'(z)}] \geq \left| \frac{zf'(z)}{f'(z)} \right|, \quad z \in \Delta \]

We mention here that the class $\text{UCV}$ was introduced (together with the class $\text{UST}$ of **uniformly starlike functions** in $A$) by A.W. Goodman ([2], [3]) who defined the uniformly starlike and convex functions as functions in $A$ with the property that the image of every circular arc contained in $\Delta$, having the center $\zeta \in \Delta$ is starlike with respect to $\zeta$ (respectively convex). These properties are expressed by using two complex variables. In 1993, Frode Ronning showed (in [8]) that $f \in \text{UCV}$ if and only if relation (1) holds. By using the theory of differential subordinations (S.S. Miller and P.T. Mocanu, [6], [7]), A. Mannino showed in 2004 ([4]) that every function $f \in \text{QUST}$ satisfies the property:
\[ \Re \sqrt{\frac{f(z)}{z}} \geq \frac{2}{3} \quad \text{in } \Delta \]
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The purpose of this paper is to generalize this last result, together with the former result of 

**A. Marx** ([5]) (with states that the principal determination of the square root of $f' \in A$ is greater than $1/2$ if $f$ is convex in $\Delta$).

We will show that:

$$f \in CV \implies \Re \sqrt[n]{f'(z)} \geq \frac{n}{n+2}$$

and

$$f \in QUST \implies \Re \sqrt[n]{f(z)} \geq \frac{n}{n+1}.$$ 

## 2 Preliminaries

For proving our principal result we will need the following definitions and results:

**Definition 1.** A function is said to be in the class $ST(\alpha)$ if and only if $f$ is in $A$ and $\Re z f'(z)/z > \alpha$ in $\Delta$.

**Lemma 1.** [1] A function $f \in A$ is convex in $\Delta$ if and only if the function $zf'(z)$ is starlike in $\Delta$.

**Lemma 1** is well-known as "Alexander’s duality theorem" and has a very simple proof based on the characterization of starlike and convex functions in the unit disc.

**Lemma 2.** [6] Let $a$ be a complex number with $\Re a > 0$ and let $
psi: \mathbb{C} \times \Delta \rightarrow \mathbb{C}$ a function satisfying:

$$\Re \psi(ix, y; z) \leq 0 \text{ in } \Delta \text{ and for all } x \text{ and } y, \text{ with } y \leq -\frac{|a - ix|^2}{2 \Re a}.$$ 

If 

$$p(z) = a + p_1 z + p_2 z^2 + \cdots \text{ is analytic in } \Delta, \text{ then} :$$

$$[\Re \psi(p(z), zp'(z); z) > 0 \text{ for all } z \in \Delta] \implies \Re p(z) > 0 \text{ in } \Delta.$$ 

Proofs of more general forms of **Lemma 2** can be found in [6] and in [7].
3 Main result

Theorem 1. If $f \in \text{ST}$ then:

$$\text{Re} \sqrt[n]{\frac{f(z)}{z}} > \frac{n}{n+2}$$

where the determination of the $n$-th root is the principal one.

Proof. Let

$$p(z) = \sqrt[n]{\frac{f(z)}{z}} - \frac{n}{n+2}$$

We have that $p(0) = 2/(n+2) > 0$ and: $f(z)/z = [p(z) = n/(n+2)]^n$.

Thus:

$$zf'(z) \over f(z) = 1 + n \frac{zp'(z)}{p(z) + \frac{n}{n+2}}$$

Denote by $\psi : \mathbb{C} \times \Delta \to \mathbb{C}, \psi(\alpha, \beta; z) = 1 + n \frac{\alpha}{\beta + n/(n+2)}$. Because $f \in \text{ST}$, we have that $\text{Re} \psi[p(z), zp'(z); z] > 0$ in $\Delta$. In order to prove that this relation implies that $\text{Re} p(z) > 0$ in $\Delta$, we will use Lemma 2 with $a = 2/(n+2)$.

$$\text{Re} \psi(ix, y; z) = \text{Re} \left[ 1 + n \frac{y}{ix + \frac{n}{n+2}} \right] = 1 + \frac{n^2(n+2)y}{n^2 + (n+2)^2x^2}$$

If

$$y \leq -\frac{|\text{Re} \ p(0) - ix|}{2 \text{Re} \ p(0)} = -\frac{4 + (n+2)^2x^2}{4(n+2)}$$

we have:

$$\text{Re} \ \psi(ix, y; z) \leq 1 - n^2 \frac{4 + (n+2)^2x^2}{4[n^2 + (n+2)^2x^2]}$$

But:

$$\frac{4 + (n+2)^2x^2}{n^2 + (n+2)^2x^2} \geq \frac{4}{n^2}$$

because the minimum of the real function $g : [0, \infty) \to \mathbb{R}$

$$g(t) = \frac{4 + (n+2)^2t}{n^2 + (n+2)^2t}$$
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is $4/n^2$. It follows that $\text{Re} \psi(ix, y; z) \leq 0$ for all real $x$ and $y \leq -\frac{\left|\text{Re} m(0) - ix\right|}{2\text{Re} m(0)}$.

By Lemma 2 we have that $\text{Re} \psi[p(z), zp'(z); z] > 0$ in $\Delta$ implies $\text{Re} p(z) > 0$ in $\Delta$, which is equivalent to: $f \in \text{ST}$ implies that $\text{Re} \sqrt[n]{f(z)/z} > n/(n + 2)$ in $\Delta$ and the theorem is proved.

**Theorem 2.** Let $f \in \text{QUST}$. Then we have:

$$\text{Re} \sqrt[n]{f(z)/z} > \frac{n}{n + 1} \text{ in } \Delta$$

where the n-th root is considered with the principal determination.

**Proof.** Let $f \in \text{QUST}$, we put, like in Theorem 1,

$$p(z) = \sqrt[n]{\frac{f(z)}{z}} - \frac{n}{n + 1}$$

It follows easily that $p(0) = 1/(n + 1) > 0$ and:

$$\frac{zf'(z)}{f(z)} = 1 + n(n + 1)\frac{zp'(z)}{n + (n + 1)p(z)}$$

Let:

$$\psi(\alpha, \beta; z) = 1 + n(n + 1)\frac{\beta}{n + (n + 1)\alpha} - \left|\frac{n(n + 1)\beta}{n + (n + 1)\alpha}\right|$$

$f \in \text{QUST}$ is equivalent to:

$$\text{Re} \psi[p(z), zp'(z); z] \geq 0 \text{ in } \delta$$

We will apply Lemma 2 for proving that this relation implies $\text{Re} p(z) > 0$ in $\Delta$, which is equivalent to: $\text{Re} \sqrt[n]{f(z)/z} > n/(n + 1)$. In order to apply Lemma 2 we have to compute $\text{Re} \psi(ix, y; z)$ and to show that this number is less or equal to zero for all real $x$ and

$$y \leq -\frac{\left|\text{Re} m(0) - ix\right|}{2\text{Re} m(0)} = -\frac{1 + (n + 1)^2x^2}{2(n + 1)}$$
A simple calculation shows that:

\[
\text{Re } \psi(ix, y; z) = 1 + n^2(n + 1) \frac{y}{n^2 + (n + 1)^2x^2} - \\
-n(n + 1) \frac{|y|}{\sqrt{n^2 + (n + 1)^2x^2}}
\]

From (2) we have that \( y = |y| \leq -\frac{1+(n+1)^2x^2}{2(n+1)} \) and thus:

\[
(3) \quad \text{Re } \psi(ix, y; z) \leq 1 - n(n + 1) \frac{1 + (n + 1)^2x^2}{2(n + 1)[n^2 + (n + 1)^2x^2]} - \\
n(n + 1) \frac{1 + (n + 1)^2x^2}{2(n + 1)\sqrt{n^2 + (n + 1)^2x^2}}
\]

Let \( g_1, g_2 : [0, \infty) \rightarrow \mathbb{R} \) given by:

\[
g_1(t) = \frac{n^2}{2} \frac{1 + (n + 1)^2t}{n^2 + (n + 1)^2t} \\
g_2(t) = \frac{n}{2} \frac{1 + (n + 1)^2t}{\sqrt{n^2 + (n + 1)^2t}}
\]

From (3) it is easy to see that \( \text{Re } \psi(ix, y; z) \leq 1 - g_1(x^2) - g_2(x^2) \). But \( g_1 \) and \( g_2 \) are increasing functions on \([0, \infty)\) and thus, \( g_1(x^2) \geq g_1(0) = 1/2 \) and \( g_2(x^2) \geq g(0) = 1/2 \) for all real \( x \). It follows that

\[
\text{Re } \psi(ix, y; z) \leq 1 - 1/2 - 1/2 = 0
\]

and the theorem is proved by applying Lemma 2.

### 4 A particular case

If we consider in Theorem 1 and in Theorem 2, \( f(z) = zg'(z) \), then the starlikeness of \( f \) is equivalent (by Lemma 1) with the convexity of \( g \) and a simple calculation shows also that \( f \in \text{QUST} \) if and only if \( g \in \text{UCV} \). We can then apply Theorem 1 and Theorem 2 to the function \( zg'(z) \) and obtain the following result:
Corollary 1. If \( g \in \text{CV} \), then we have:

\[
\Re \sqrt[n]{g'(z)} \geq \frac{n}{n + 2}
\]

and if \( g \in \text{UCV} \) we have

\[
\Re \sqrt[n]{g'(z)} \geq \frac{n}{n + 1}
\]

where the \( n \)-th roots are considered with their principal determinations.

References


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