

ON THE SINGULAR TWO-PARAMETER EIGENVALUE PROBLEM*

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Abstract. In the 1960s, Atkinson introduced an abstract algebraic setting for multiparameter eigenvalue problems. He showed that a nonsingular multiparameter eigenvalue problem is equivalent to the associated system of generalized eigenvalue problems. Many theoretical results and numerical methods for nonsingular multiparameter eigenvalue problems are based on this relation. In this paper, the above relation to singular two-parameter eigenvalue problems is extended, and it is shown that the simple finite regular eigenvalues of a two-parameter eigenvalue problem and the associated system of generalized eigenvalue problems agree. This enables one to solve a singular two-parameter eigenvalue problem by computing the common regular eigenvalues of the associated system of two singular generalized eigenvalue problems.

Key words. Singular two-parameter eigenvalue problem, Operator determinant, Kronecker canonical form, Minimal reducing subspace.

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1. Introduction. We consider *the algebraic two-parameter eigenvalue problem*

$$(1.1) \quad \begin{aligned} W_1(\lambda, \mu)x_1 &:= (A_1 + \lambda B_1 + \mu C_1)x_1 = 0, \\ W_2(\lambda, \mu)x_2 &:= (A_2 + \lambda B_2 + \mu C_2)x_2 = 0, \end{aligned}$$

where $A_i, B_i,$ and C_i are $n_i \times n_i$ matrices over \mathbb{C} , $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$. A pair (λ, μ) is an *eigenvalue* if it satisfies (1.1) for nonzero vectors x_1, x_2 , and the tensor product $x_1 \otimes x_2$ is the corresponding (*right*) *eigenvector*. Similarly, $y_1 \otimes y_2$ is the corresponding *left eigenvector* if $y_i \neq 0$ and $y_i^* W_i(\lambda, \mu) = 0$ for $i = 1, 2$.

The eigenvalues of (1.1) are the roots of the following system of two bivariate polynomials

$$(1.2) \quad \begin{aligned} p_1(\lambda, \mu) &:= \det(W_1(\lambda, \mu)) = 0, \\ p_2(\lambda, \mu) &:= \det(W_2(\lambda, \mu)) = 0. \end{aligned}$$

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A two-parameter eigenvalue problem can be expressed as two coupled generalized eigenvalue problems. On the tensor product space $S := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of the dimension $N := n_1 n_2$ we define *operator determinants*

$$(1.3) \quad \begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2, \\ \Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2, \end{aligned}$$

for details see, e.g., [2]. The problem (1.1) is then related to a coupled pair of generalized eigenvalue problems

$$(1.4) \quad \begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned}$$

for decomposable tensors $z \in S$, $z = x \otimes y$. The precise nature of this relation will be discussed in this paper.

Usually we assume that the two-parameter eigenvalue problem (1.1) is *nonsingular*, i.e., the corresponding operator determinant Δ_0 is nonsingular. In this case (see, e.g., [2]), the matrices $\Delta_0^{-1} \Delta_1$ and $\Delta_0^{-1} \Delta_2$ commute and the eigenvalues of (1.1) agree with the eigenvalues of (1.4). By applying this relation, a nonsingular two-parameter eigenvalue problem can be numerically solved using standard tools for the generalized eigenvalue problems, for an algorithm see, e.g., [9].

Let us remark that Atkinson [2] uses the homogeneous formulation of the problem

$$(1.5) \quad \begin{aligned} (\eta_0 A_1 + \eta_1 B_1 + \eta_2 C_1) x_1 &= 0, \\ (\eta_0 A_2 + \eta_1 B_2 + \eta_2 C_2) x_2 &= 0, \end{aligned}$$

where $(\eta_0, \eta_1, \eta_2) \neq (0, 0, 0)$. The homogeneous formulation of the problem (1.5) is nonsingular if there exists a nonsingular linear combination $\Delta = \alpha_0 \Delta_0 + \alpha_1 \Delta_1 + \alpha_2 \Delta_2$. Then (see [2]) the matrices $\Delta^{-1} \Delta_0$, $\Delta^{-1} \Delta_1$, and $\Delta^{-1} \Delta_2$ commute and we get η_0 , η_1 , and η_2 from the joint generalized eigenvalue problems $\Delta_0 z = \eta_0 \Delta z$, $\Delta_1 z = \eta_1 \Delta z$, and $\Delta_2 z = \eta_2 \Delta z$. An eigenvalue (η_0, η_1, η_2) of (1.5) with $\eta_0 \neq 0$ gives a finite eigenvalue $(\lambda, \mu) = (\eta_1/\eta_0, \eta_2/\eta_0)$ of (1.1).

Several applications lead to singular two-parameter eigenvalue problems, where Δ_0 is singular and (1.4) is a pair of singular generalized eigenvalue problems. Here we assume that the problem is singular in the homogeneous setting as well, i.e., $\det(\alpha_0 \Delta_0 + \alpha_1 \Delta_1 + \alpha_2 \Delta_2) = 0$ for all $(\alpha_0, \alpha_1, \alpha_2)$. Two examples of singular problems are the model updating [4] and the quadratic two-parameter eigenvalue problem [13]. Apart from a few theoretical results and numerical methods in [4] and [13], which only cover very specific examples, the singular two-parameter eigenvalue problem is

still open. We extend Atkinson's results from [2] and show that simple eigenvalues of the singular two-parameter eigenvalue problem (1.1) can be computed from the eigenvalues of the corresponding pair of singular generalized eigenvalue problems (1.4). This opens new possibilities in the study of singular two-parameter eigenvalue problems. The new results justify that the numerical method presented in [13] can be applied not only to the linearization of a quadratic two-parameter eigenvalue problem but also to a general singular two-parameter eigenvalue problem.

DEFINITION 1.1. The *normal rank* of a two-parameter matrix pencil $W_i(\lambda, \mu)$ is

$$\text{nrnk}(W_i(\lambda, \mu)) = \max_{\lambda, \mu \in \mathbb{C}} \text{rank}(W_i(\lambda, \mu))$$

for $i = 1, 2$. A pair $(\lambda_0, \mu_0) \in \mathbb{C}^2$ is a *finite regular eigenvalue* of the two-parameter eigenvalue problem (1.1) if $\text{rank}(W_i(\lambda_0, \mu_0)) < \text{nrnk}(W_i(\lambda, \mu))$ for $i = 1, 2$. The *geometric multiplicity* of the eigenvalue (λ_0, μ_0) is equal to

$$\prod_{i=1}^2 \left(\text{nrnk}(W_i(\lambda, \mu)) - \text{rank}(W_i(\lambda_0, \mu_0)) \right).$$

Throughout this paper, we assume that the two-parameter eigenvalue problem (1.1) is *regular*, which means that both matrix pencils $W_1(\lambda, \mu)$ and $W_2(\lambda, \mu)$ have full normal rank, i.e., $\text{nrnk}(W_i(\lambda, \mu)) = n_i$ for $i = 1, 2$. This is equivalent to the condition that none of the polynomials p_1 and p_2 is identically zero. We also assume that p_1 and p_2 do not have a nontrivial common divisor (scalars are regarded as trivial common divisors), because this would lead to infinitely many eigenvalues. If the greatest common divisor of p_1 and p_2 is a scalar, then (1.1) has (counting with multiplicities) $k \leq N$ finite regular eigenvalues, where $k \leq \text{rank}(\Delta_0)$.

In the next section, we introduce the Kronecker canonical form and other auxiliary results. In Section 3, we show that all simple eigenvalues of a singular two-parameter eigenvalue problem (1.1) agree with the finite regular eigenvalues of the associated pair of generalized eigenvalue problems (1.4). In Section 4, we give examples of small two-parameter eigenvalue problems that support the theory, and in the final section, we review how to numerically solve a singular two-parameter eigenvalue problem using the algorithm for the computation of the common regular subspace of a singular matrix pencil, presented in [13].

2. Auxiliary results. In this section, we review the Kronecker canonical form and Kronecker chains of a matrix pencil. More about the Kronecker canonical form and its numerical computation can be found in, e.g., [6], [7], [8], and [16], for Kronecker chains see, e.g., [12].

DEFINITION 2.1. Let $A - \lambda B \in \mathbb{C}^{m \times n}$ be a matrix pencil. Then there exist nonsingular matrices $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ such that

$$P^{-1}(A - \lambda B)Q = \tilde{A} - \lambda \tilde{B} = \text{diag}(A_1 - \lambda B_1, \dots, A_k - \lambda B_k)$$

is in *Kronecker canonical form*. Each block $A_i - \lambda B_i$, $i = 1, \dots, k$, has one of the following forms: $J_d(\alpha)$, N_d , L_d , or L_d^T , where the matrices

$$J_d(\alpha) = \begin{bmatrix} \alpha - \lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \alpha - \lambda \end{bmatrix} \in \mathbb{C}^{d \times d}, \quad N_d = \begin{bmatrix} 1 & -\lambda & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -\lambda \\ & & & & 1 \end{bmatrix} \in \mathbb{C}^{d \times d},$$

$$L_d = \begin{bmatrix} -\lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\lambda & 1 \end{bmatrix} \in \mathbb{C}^{d \times (d+1)}, \quad L_d^T = \begin{bmatrix} -\lambda & & & & \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & -\lambda & \\ & & & & 1 \end{bmatrix} \in \mathbb{C}^{(d+1) \times d},$$

represent a finite regular block, an infinite regular block, a right singular block, and a left singular block, respectively. To each Kronecker block we associate a *Kronecker chain* of linearly independent vectors as follows:

- a) A finite regular block $J_d(\alpha)$ is associated with vectors u_1, \dots, u_d that satisfy

$$\begin{aligned} (A - \alpha B)u_1 &= 0, \\ (A - \alpha B)u_{i+1} &= Bu_i, \quad i = 1, \dots, d-1. \end{aligned}$$

- b) An infinite regular block N_d is associated with vectors u_1, \dots, u_d that satisfy

$$\begin{aligned} Bu_1 &= 0, \\ Bu_{i+1} &= Au_i, \quad i = 1, \dots, d-1. \end{aligned}$$

- c) A right singular block L_d is associated with vectors u_1, \dots, u_{d+1} that satisfy

$$\begin{aligned} Bu_1 &= 0, \\ Bu_{i+1} &= Au_i, \quad i = 1, \dots, d, \\ 0 &= Au_{d+1}. \end{aligned}$$

- d) For $d \geq 1$, a left singular block L_d^T is associated with vectors u_1, \dots, u_d that satisfy

$$Bu_i = Au_{i+1}, \quad i = 1, \dots, d-1.$$

The union of the Kronecker chains for all Kronecker blocks is a basis for \mathbb{C}^n . We say that a subspace $\mathcal{M} \subset \mathbb{C}^n$ is a *reducing subspace* for the pencil $A - \lambda B$ if $\dim(\mathcal{M} + B\mathcal{M}) = \dim(\mathcal{M}) - s$, where $s \geq 0$ is the number of the right singular blocks L_d in the Kronecker canonical form for $A - \lambda B$. The vectors from the Kronecker chains of all right singular blocks L_d form a basis for the *minimal reducing subspace* $\mathcal{R}(A, B)$, which is a subset of all reducing subspaces. The minimal reducing subspace is unique and can be numerically computed in a stable way from the generalized upper triangular form (GUPTRI), see, e.g., [6, 7].

DEFINITION 2.2. The *normal rank* of a square matrix pencil $A - \lambda B$ is

$$\text{nrnk}(A - \lambda B) = \max_{\lambda \in \mathbb{C}} \text{rank}(A - \lambda B).$$

A scalar $\lambda_0 \in \mathbb{C}$ is a *finite regular eigenvalue* if $\text{rank}(A - \lambda_0 B) < \text{nrnk}(A - \lambda B)$, its *geometric multiplicity* is $\text{nrnk}(A - \lambda B) - \text{rank}(A - \lambda_0 B)$. Let $(A - \lambda_0 B)z = 0$, where $z \neq 0$. If z does not belong to the minimal reducing subspace $\mathcal{R}(A, B)$ then z is a *regular eigenvector*, otherwise, z is a *singular eigenvector*.

It is trivial to construct a basis for the kernel of the tensor product $A \otimes D$ from the kernels of A and D . The task is much harder if we take a difference of two tensor products, which is the form of the operator determinants (1.3). Košir shows in [12] that the kernel of an operator determinant $\Delta = A \otimes D - B \otimes C$ can be constructed from the Kronecker chains of matrix pencils $A - \lambda B$ and $C - \mu D$.

THEOREM 2.3 ([12, Theorem 4]). *A basis for the kernel of $\Delta = A \otimes D - B \otimes C$ is the union of sets of linearly independent vectors associated with the pairs of Kronecker blocks of the following types:*

- a) $(J_{d_1}(\alpha_1), J_{d_2}(\alpha_2))$, where $\alpha_1 = \alpha_2$,
- b) (N_{d_1}, N_{d_2}) ,
- c) (N_{d_1}, L_{d_2}) ,
- d) (L_{d_1}, N_{d_2}) ,
- e) $(L_{d_1}, J_{d_2}(\alpha))$,
- f) $(J_{d_1}(\alpha), L_{d_2})$,
- g) (L_{d_1}, L_{d_2}) ,
- h) $(L_{d_1}, L_{d_2}^T)$, where $d_1 < d_2$,
- i) $(L_{d_1}^T, L_{d_2})$, where $d_1 > d_2$,

where the left block of each pair belongs to the pencil $A - \lambda B$ and the right block belongs to the pencil $C - \mu D$.

For each pair of Kronecker blocks that satisfies a), b), c), or d) we can construct an associated set of linearly independent vectors z_1, \dots, z_d in the kernel of Δ as follows:

- a), b) Let the vectors u_1, \dots, u_{d_1} form a Kronecker chain associated with the block

$J_{d_1}(\alpha_1)$ (or N_{d_1}) of the pencil $A - \lambda B$ and let the vectors v_1, \dots, v_{d_2} form a Kronecker chain associated with the block $J_{d_2}(\alpha_2)$ (or N_{d_2}) of the pencil $C - \mu D$. Then $d = \min(d_1, d_2)$ and

$$z_j = \sum_{i=1}^j u_i \otimes v_{j+1-i}, \quad j = 1, \dots, d.$$

c) Let the vectors u_1, \dots, u_{d_1} form a Kronecker chain associated with the block N_{d_1} of the pencil $A - \lambda B$ and let the vectors v_1, \dots, v_{d_2+1} form a Kronecker chain associated with the block L_{d_2} of the pencil $C - \mu D$. Then $d = d_1$ and

$$z_j = \sum_{i=\max(1, j-d_2)}^j u_i \otimes v_{j+1-i}, \quad j = 1, \dots, d.$$

d) Let the vectors u_1, \dots, u_{d_1+1} form a Kronecker chain associated with the block L_{d_1} of the pencil $A - \lambda B$ and let the vectors v_1, \dots, v_{d_2} form a Kronecker chain associated with the block N_{d_2} of the pencil $C - \mu D$. Then $d = d_2$ and

$$z_j = \sum_{i=1}^{\min(d_1+1, j)} u_i \otimes v_{j+1-i}, \quad j = 1, \dots, d.$$

In the above theorem, we omitted the constructions of vectors in the kernel for all pairs of Kronecker blocks where details are not relevant for our case. For a complete description, see [12]. A similar technique is used to describe the kernels of generalized Sylvester operators in [5].

3. Delta matrices and simple eigenvalues. In the nonsingular case, the eigenvalues of (1.1) agree with the eigenvalues of the associated pair of generalized eigenvalue problems (1.4), see, e.g., [2]. In this section, we show that in a similar way the finite regular eigenvalues of (1.1) are related to the finite regular eigenvalues of (1.4).

DEFINITION 3.1. A pair $(\lambda_0, \mu_0) \in \mathbb{C}^2$ is a *finite regular eigenvalue* of the matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if the following is true:

- a) λ_0 is a finite regular eigenvalue of $\Delta_1 - \lambda\Delta_0$,
- b) μ_0 is a finite regular eigenvalue of $\Delta_2 - \mu\Delta_0$,
- c) there exists a common regular eigenvector z , i.e., $z \neq 0$ such that $(\Delta_1 - \lambda_0\Delta_0)z = 0$, $(\Delta_2 - \mu_0\Delta_0)z = 0$, and $z \notin \mathcal{R}(\Delta_i, \Delta_0)$ for $i = 1, 2$.

The *geometric multiplicity* of (λ_0, μ_0) is $\dim(\mathcal{N}) - \dim(\mathcal{N} \cap (\mathcal{R}(\Delta_1, \Delta_0) \cup \mathcal{R}(\Delta_2, \Delta_0)))$, where $\mathcal{N} = \text{Ker}(\Delta_1 - \lambda_0\Delta_0) \cap \text{Ker}(\Delta_2 - \mu_0\Delta_0)$.

In order to obtain a general result, instead of the linear two-parameter eigenvalue problem (1.1), we consider a nonlinear two-parameter eigenvalue problem:

$$(3.1) \quad \begin{aligned} T_1(\lambda, \mu)x_1 &= 0 \\ T_2(\lambda, \mu)x_2 &= 0, \end{aligned}$$

where $T_i(\cdot, \cdot) : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^{n_i \times n_i}$ is differentiable for $i = 1, 2$. If (3.1) is satisfied for nonzero vectors x_1 and x_2 , then (λ, μ) is an eigenvalue and $x_1 \otimes x_2$ is the corresponding right eigenvector. The corresponding left eigenvector is $y_1 \otimes y_2$ such that $y_i \neq 0$ and $y_i^* T_i(\lambda, \mu) = 0$ for $i = 1, 2$.

If y and x are the left and the right eigenvector of a simple eigenvalue λ_0 of a nonlinear eigenvalue problem $T(\lambda)x = 0$, where T is differentiable, then it is well-known that $y^* T'(\lambda)x \neq 0$, see, e.g., [1, 14]. The following proposition generalizes this relation to the nonlinear two-parameter eigenvalue problem. The proof is based on the one-parameter version from [15].

PROPOSITION 3.2. *Let (λ_0, μ_0) be an algebraically and geometrically simple eigenvalue of the nonlinear two-parameter eigenvalue problem (3.1) and let $x_1 \otimes x_2$ and $y_1 \otimes y_2$ be the corresponding right and left eigenvector. Then the matrix*

$$M_0 := \begin{bmatrix} y_1^* \frac{\partial T_1}{\partial \lambda}(\lambda_0, \mu_0)x_1 & y_1^* \frac{\partial T_1}{\partial \mu}(\lambda_0, \mu_0)x_1 \\ y_2^* \frac{\partial T_2}{\partial \lambda}(\lambda_0, \mu_0)x_2 & y_2^* \frac{\partial T_2}{\partial \mu}(\lambda_0, \mu_0)x_2 \end{bmatrix}$$

is nonsingular.

Proof. For $i = 1, 2$ we define the $n_i \times n_i$ matrix

$$S_i(\lambda, \mu) = \begin{bmatrix} T_i(\lambda, \mu) & y_i \\ x_i^* & 0 \end{bmatrix}.$$

Since (λ_0, μ_0) is a simple eigenvalue, $\text{rank}(T_i(\lambda_0, \mu_0)) = n_i - 1$. From $T_i(\lambda_0, \mu_0)x_i = 0$ we know that the first n_i columns of $S_i(\lambda_0, \mu_0)$ are linearly independent. In addition, the first n_i columns of $S_i(\lambda_0, \mu_0)$ are orthogonal to the last column because of $y_i^* T_i(\lambda_0, \mu_0) = 0$. It follows that $S_i(\lambda_0, \mu_0)$ is a nonsingular matrix.

Let us denote the element in the lower right corner of $S_i(\lambda, \mu)^{-1}$ by $\alpha_i(\lambda, \mu)$, i.e.,

$$(3.2) \quad \alpha_i(\lambda, \mu) = e_{n_i+1}^T S_i(\lambda, \mu)^{-1} e_{n_i+1}.$$

Let $r_i(\lambda, \mu) = \det(T_i(\lambda, \mu))$ and $q_i(\lambda, \mu) = \det(S_i(\lambda, \mu))$. It follows from a well-known relation between the element of the inverse matrix and the corresponding cofactor that in a small neighborhood of (λ_0, μ_0) we have

$$(3.3) \quad r_i(\lambda, \mu) = \alpha_i(\lambda, \mu) \cdot q_i(\lambda, \mu).$$

We know that $r_i(\lambda_0, \mu_0) = 0$ and $q_i(\lambda_0, \mu_0) \neq 0$, therefore $\alpha_i(\lambda_0, \mu_0) = 0$ for $i = 1, 2$. By differentiating the expression (3.2) at (λ_0, μ_0) we get

$$(3.4) \quad \begin{aligned} \frac{\partial \alpha_i}{\partial \lambda}(\lambda_0, \mu_0) &= -e_{n_i+1}^T S_i(\lambda_0, \mu_0)^{-1} \frac{\partial S_i}{\partial \lambda}(\lambda_0, \mu_0) S_i(\lambda_0, \mu_0)^{-1} e_{n_i+1} \\ &= -[y_i^* \quad 0] \begin{bmatrix} \frac{\partial T_i}{\partial \lambda}(\lambda_0, \mu_0) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ 0 \end{bmatrix} = -y_i^* \frac{\partial T_i}{\partial \lambda}(\lambda_0, \mu_0) x_i \end{aligned}$$

and, in a similar way,

$$(3.5) \quad \frac{\partial \alpha_i}{\partial \mu}(\lambda_0, \mu_0) = -y_i^* \frac{\partial T_i}{\partial \mu}(\lambda_0, \mu_0) x_i.$$

The partial derivatives of the expression (3.3) at (λ_0, μ_0) are

$$(3.6) \quad \frac{\partial r_i}{\partial \lambda}(\lambda_0, \mu_0) = \frac{\partial \alpha_i}{\partial \lambda}(\lambda_0, \mu_0) \cdot q_i(\lambda_0, \mu_0)$$

$$(3.7) \quad \frac{\partial r_i}{\partial \mu}(\lambda_0, \mu_0) = \frac{\partial \alpha_i}{\partial \mu}(\lambda_0, \mu_0) \cdot q_i(\lambda_0, \mu_0)$$

for $i = 1, 2$. We combine the equations (3.4), (3.5), (3.6), and (3.7) into

$$(3.8) \quad \begin{bmatrix} \frac{\partial r_1}{\partial \lambda}(\lambda_0, \mu_0) & \frac{\partial r_1}{\partial \mu}(\lambda_0, \mu_0) \\ \frac{\partial r_2}{\partial \lambda}(\lambda_0, \mu_0) & \frac{\partial r_2}{\partial \mu}(\lambda_0, \mu_0) \end{bmatrix} = - \begin{bmatrix} q_1(\lambda_0, \mu_0) & \\ & q_2(\lambda_0, \mu_0) \end{bmatrix} M_0.$$

The matrix on the left hand side of (3.8) is nonsingular as it is the Jacobian matrix of the nonlinear system $r_1(\lambda, \mu) = 0, r_2(\lambda, \mu) = 0$ at a simple root $(\lambda, \mu) = (\lambda_0, \mu_0)$. This implies that M_0 is nonsingular. \square

COROLLARY 3.3. *Let (λ_0, μ_0) be an algebraically simple eigenvalue of the two-parameter eigenvalue problem (1.1) and let $x_1 \otimes x_2$ and $y_1 \otimes y_2$ be the corresponding right and left eigenvector. It follows that*

$$(3.9) \quad (y_1 \otimes y_2)^* \Delta_0(x_1 \otimes x_2) = \begin{vmatrix} y_1^* B_1 x_1 & y_1^* C_1 x_1 \\ y_2^* B_2 x_2 & y_2^* C_2 x_2 \end{vmatrix} \neq 0.$$

Let us remark that the result in Corollary 3.3 was obtained for the nonsingular multiparameter eigenvalue problem by Košir in [11, Lemma 3]. The connection (3.8) between the Jacobian matrix of the polynomial system (1.2) and the matrix

$$\begin{bmatrix} y_1^* B_1 x_1 & y_1^* C_1 x_1 \\ y_2^* B_2 x_2 & y_2^* C_2 x_2 \end{bmatrix}$$

was established for the nonsingular right definite two-parameter eigenvalue problem in [10, Proposition 13].

LEMMA 3.4. *Let λ_0 be an eigenvalue of the matrix pencil $A - \lambda B$ with the corresponding right and left eigenvector x and y , respectively. If*

$$(3.10) \quad y^* Bx \neq 0$$

then λ_0 is a finite regular eigenvalue.

Proof. It suffices to show that the eigenvector x is not part of the singular subspace of $A - \lambda B$, which is composed of vector polynomials $u(\lambda)$ such that $(A - \lambda B)u(\lambda) = 0$ for all λ .

Suppose that there exists a polynomial $u(\lambda)$ such that

$$(3.11) \quad (A - \lambda B)u(\lambda) = 0$$

for all λ and $u(\lambda_0) = x$. If we differentiate (3.11) at $\lambda = \lambda_0$ then we obtain

$$(A - \lambda_0 B)u'(\lambda_0) - Bx = 0.$$

When we multiply this equality by y^* we get $y^* Bx = 0$, which contradicts the assumption (3.10). So, such a polynomial $u(\lambda)$ does not exist and x is not in the singular subspace. Therefore, the rank of the matrix pencil $A - \lambda B$ drops at $\lambda = \lambda_0$ and λ_0 is a regular eigenvalue. It follows from $Bx \neq 0$ that λ_0 is a finite eigenvalue. \square

THEOREM 3.5. *If (λ_0, μ_0) is an algebraically simple finite regular eigenvalue of a regular two-parameter eigenvalue problem (1.1) then (λ_0, μ_0) is a finite regular eigenvalue of the associated pair of generalized eigenvalue problems (1.4).*

Proof. Let (λ_0, μ_0) be a finite regular eigenvalue of (1.1) and let $z = x_1 \otimes x_2$ and $w = y_1 \otimes y_2$ be the corresponding right and left eigenvector, respectively. It follows from Corollary 3.3 that $w^* \Delta_0 z \neq 0$. Now we can apply Lemma 3.4 to conclude that λ_0 is a finite regular eigenvalue of $\Delta_1 - \lambda \Delta_0$.

We can show the same for the eigenvalue μ_0 of the matrix pencil $\Delta_2 - \mu \Delta_0$. It follows that (λ_0, μ_0) is a finite regular eigenvalue of pencils $\Delta_1 - \lambda \Delta_0$ and $\Delta_2 - \mu \Delta_0$ with the common regular eigenvector z . \square

COROLLARY 3.6. *If all finite eigenvalues of a regular two-parameter eigenvalue problem (1.1) are algebraically simple, then all finite eigenvalues of (1.1) are finite regular eigenvalues of the associated pair of generalized eigenvalue problems (1.4).*

In order to establish a bidirectional link between the eigenvalues of the two-parameter eigenvalue problem (1.1) and the eigenvalues of the associated pair of generalized eigenvalue problems (1.4), we have to prove the relation in the opposite direction as well. We do this in the following theorem.

THEOREM 3.7. *Let us assume that all finite eigenvalues of a regular two-parameter eigenvalue problem (1.1) are algebraically simple. Let (λ_0, μ_0) be a finite regular eigenvalue of the associated pair of generalized eigenvalue problems (1.4). Then (λ_0, μ_0) is a finite regular eigenvalue of the regular two-parameter eigenvalue problem (1.1).*

Proof. Let $z \in \text{Ker}(\Delta_1 - \lambda_0 \Delta_0) \cap \text{Ker}(\Delta_2 - \mu_0 \Delta_0)$ be a common regular eigenvector for the eigenvalue (λ_0, μ_0) . We can write

$$(3.12) \quad \Delta_1 - \lambda_0 \Delta_0 = W_1(\lambda_0, 0) \otimes C_2 - C_1 \otimes W_2(\lambda_0, 0).$$

It follows from Theorem 2.3 that z is a linear combination of vectors associated with appropriate pairs of Kronecker blocks of pencils $W_1(\lambda_0, 0) - \alpha_1 C_1$ and $W_2(\lambda_0, 0) - \alpha_2 C_2$. Since z is a common regular eigenvector, at least one of these vectors must not belong to the right singular subspace of the pencil $\Delta_2 - \mu \Delta_0$ (and the same for the pencil $\Delta_1 - \lambda \Delta_0$).

First, we show that pairs of Kronecker blocks of the types $(L_{d_1}, L_{d_2}^T)$, $(L_{d_1}^T, L_{d_2})$, and (L_{d_1}, L_{d_2}) do not appear. Namely, any of the above combinations implies that $p_i(\lambda_0, \alpha_i) = \det(W_i(\lambda_0, -\alpha_i)) = 0$ for all α_i and $i = 1, 2$. This contradicts the assumption that the problem (1.1) has finitely many eigenvalues, which are solutions of the system $p_1(\lambda, \mu) = 0, p_2(\lambda, \mu) = 0$.

Next, we consider the possibility that z is a linear combination of vectors that belong to pairs of Kronecker blocks of the types (N_{d_1}, N_{d_2}) , (N_{d_1}, L_{d_2}) , and (L_{d_1}, N_{d_2}) . Suppose that the pencil $W_i(\lambda_0, 0) - \alpha_i C_i$ has a Kronecker block N_{d_i} for $i = 1, 2$. The two Kronecker blocks are associated with the Kronecker chains u_1, \dots, u_{d_1} and v_1, \dots, v_{d_2} such that

$$C_1 u_1 = 0, \quad C_1 u_2 = W_1(\lambda_0, 0) u_1, \quad \dots, \quad C_1 u_{d_1} = W_1(\lambda_0, 0) u_{d_1-1},$$

and

$$C_2 v_1 = 0, \quad C_2 v_2 = W_2(\lambda_0, 0) v_1, \quad \dots, \quad C_2 v_{d_2} = W_2(\lambda_0, 0) v_{d_2-1}.$$

It follows from Theorem 2.3 that the vectors z_1, \dots, z_d , where

$$z_j = \sum_{i=1}^j u_i \otimes v_{j+1-i}$$

and $d = \min(d_1, d_2)$, are in the basis for $\text{Ker}(\Delta_1 - \lambda_0 \Delta_0)$. For $z_1 = u_1 \otimes v_1$ it is easy to see that

$$(3.13) \quad \Delta_1 z_1 = \Delta_0 z_1 = 0.$$

Therefore, z_1 is not a regular eigenvector of $\Delta_1 - \lambda\Delta_0$ at $\lambda = \lambda_0$. If $d > 1$ then

$$\begin{aligned}
 \Delta_0 z_j &= (B_1 \otimes C_2 - C_1 \otimes B_2) \sum_{i=1}^j u_i \otimes v_{j+1-i} \\
 (3.14) \quad &= B_1 \otimes W_2(\lambda_0, 0) \sum_{i=1}^{j-1} u_i \otimes v_{j-i} - W_1(\lambda_0, 0) \otimes B_2 \sum_{i=2}^j u_{i-1} \otimes v_{j+1-i} \\
 &= \left(B_1 \otimes W_2(\lambda_0, 0) - W_1(\lambda_0, 0) \otimes B_2 \right) \sum_{i=1}^{j-1} u_i \otimes v_{j-i} = -\Delta_2 z_{j-1}.
 \end{aligned}$$

In a similar way, one can show that the equations (3.13) and (3.14) are also true for a set of vectors z_1, \dots, z_d constructed by Theorem 2.3 from the Kronecker chains for a pair of Kronecker blocks of the type (N_{d_1}, L_{d_2}) or (L_{d_1}, N_{d_2}) .

Suppose that the basis for the kernel of $\Delta_1 - \lambda_0\Delta_0$ is a union of sets of vectors that belong to m pairs of Kronecker blocks of the types (N_{d_1}, N_{d_2}) , (N_{d_1}, L_{d_2}) , and (L_{d_1}, N_{d_2}) only. Then it follows from (3.13) and (3.14) that there exist vectors z_{k1}, \dots, z_{kd_k} for $k = 1, \dots, m$ such that

$$(3.15) \quad \Delta_0 z_{k1} = 0$$

and

$$(3.16) \quad \Delta_0 z_{kj} = -\Delta_2 z_{k,j-1}$$

for $j = 2, \dots, d_k$. By Theorem 2.3, we can expand the common regular eigenvector z in this basis as

$$z = \sum_{j=1}^m \sum_{k=1}^{d_j} \xi_{jk} z_{jk}.$$

Using the relation (3.16), we define a chain of vectors $z^{(0)}, \dots, z^{(D)}$ such that

$$\begin{aligned}
 (\Delta_2 - \mu_0\Delta_0)z &= \Delta_2 \left(\sum_{j=1}^m \left(\sum_{k=1}^{d_j} \xi_{jk} z_{jk} + \mu_0 \sum_{k=2}^{d_j} \xi_{jk} z_{j,k-1} \right) \right) = \Delta_2 z^{(0)} = 0 \\
 \Delta_0 z^{(0)} &= -\Delta_2 \left(\sum_{j=1}^m \left(\sum_{k=2}^{d_j} \xi_{jk} z_{j,k-1} + \mu_0 \sum_{k=3}^{d_j} \xi_{jk} z_{j,k-2} \right) \right) = -\Delta_2 z^{(1)} \\
 &\vdots \\
 \Delta_0 z^{(D-1)} &= -\Delta_2 \left(\sum_{j=1}^m \left(\sum_{k=D+1}^{d_j} \xi_{jk} z_{j,k-D} + \mu_0 \sum_{k=D+2}^{d_j} \xi_{jk} z_{j,k-D-1} \right) \right) = -\Delta_2 z^{(D)},
 \end{aligned}$$

where $D = \max\{k - 1 : \xi_{jk} \neq 0, j = 1, \dots, m, k = 1, \dots, d_j\}$. The chain ends with $\Delta_0 z^{(D)} = 0$, which follows from (3.15) and

$$z^{(D)} = \sum_{\substack{j=1 \\ d_j \geq D+1}}^m \xi_{jD} z_{j1}.$$

The relations $\Delta_2 z^{(0)} = 0, \Delta_0 z^{(0)} = -\Delta_2 z^{(1)}, \dots, \Delta_0 z^{(D-1)} = -\Delta_2 z^{(D)}, \Delta_0 z^{(D)} = 0$ show that $z^{(0)}, \dots, z^{(D)}$ belong to the right singular subspace of the pencil $\Delta_2 - \mu\Delta_0$ (see, e.g., [8, Section 12.3]). It follows that z , which is a linear combination of the vectors $z^{(0)}, \dots, z^{(D)}$, belongs to the singular part of $\Delta_2 - \mu\Delta_0$. We conclude that by vectors solely from the combinations of the types $(N_{d_1}, N_{d_2}), (N_{d_1}, L_{d_2}),$ and (L_{d_1}, N_{d_2}) it is not possible to write down the common regular eigenvector z .

From the above discussion, we see that the only option for the existence of a common regular eigenvector is that there exists a Kronecker pair of one of the following types $(J_{d_1}(\alpha), J_{d_2}(\alpha)), (L_{p_1}, J_{p_2}(\alpha)),$ or $(J_{p_1}(\alpha), L_{p_2})$. If such a pair exists, then $\det(W_i(\lambda_0, -\alpha)) = 0$ for $i = 1, 2$ and $(\lambda_0, -\alpha)$ is a finite eigenvalue of the two-parameter eigenvalue problem (1.1), where we denote by $x_1 \otimes x_2$ the corresponding eigenvector. As we assume that all finite eigenvalues of (1.1) are algebraically simple, it follows from Theorem 3.5 that $(\lambda_0, -\alpha)$ is an eigenvalue of the associated pair of generalized eigenvalue problems (1.4) with the common regular eigenvector $x_1 \otimes x_2$.

So, there exists an eigenvalue $(\lambda_0, -\alpha)$ of (1.1) with the corresponding eigenvector $x_1 \otimes x_2$ such that $z = \xi x_1 \otimes x_2 + s$, where $\xi \neq 0$ and the vectors s and $x_1 \otimes x_2$ are linearly independent. Suppose that $-\alpha \neq \mu_0$. We know that z is a regular eigenvector for the eigenvalue μ_0 of the pencil $\Delta_2 - \mu\Delta_0$. As such, z can not have a nonzero component in the direction of $x_1 \otimes x_2$, which is a regular eigenvector for the eigenvalue $-\alpha \neq \mu_0$ of $\Delta_2 - \mu\Delta_0$. Therefore, $-\alpha = \mu_0$ and (λ_0, μ_0) is a finite regular eigenvalue of (1.1). \square

From the above proof some interesting properties of the pencil $\Delta_1 - \lambda\Delta_0$ (same applies to $\Delta_2 - \mu\Delta_0$) can be deduced. We collect them in the following corollary.

COROLLARY 3.8.

- a) If the pencil $\Delta_1 - \lambda\Delta_0$ is singular, then it contains at least one block L_0 .
- b) Suppose that λ_0 is not a regular eigenvalue of the pencil $\Delta_1 - \lambda\Delta_0$. Let $W_i(\lambda_0, 0) - \alpha_i C_i$ for $i = 1, 2$ be a regular pencil such that its Kronecker canonical form contains infinite regular blocks $N_{d_1}, \dots, N_{d_{k_i}}$. Then

$$(3.17) \quad \text{rank}(\Delta_1 - \lambda_0\Delta_0) = N - \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \min(p_i, p_j)$$

and the Kronecker canonical form of the pencil $\Delta_1 - \lambda\Delta_0$ contains at least $k_1 k_2$ blocks L_0 .

From (3.17) one can compute the normal rank of the pencil $\Delta_1 - \lambda\Delta_0$ without working with the large matrices Δ_0 and Δ_1 explicitly.

REMARK 3.9. A natural question is whether we can extend the relations in Theorem 3.5 and Theorem 3.7 to multiparameter eigenvalue problems with more than two parameters?

One direction is simple. It is straightforward to extend Proposition 3.2 and Corollary 3.3 to cover problems with more than two parameters. This allows us to generalize Theorem 3.5 and show that simple eigenvalues of a singular multiparameter eigenvalue problem are common regular eigenvalues of the associated system of singular generalized eigenvalue problems.

But, it is not clear how to prove the connection in the other direction. The proof of Theorem 3.7 relies on Theorem 2.3 which is only available for 2×2 operator determinants. So, in order to generalize Theorem 3.7 to more than two parameters, a different approach is required and this problem is still open.

4. Examples. In this section we present some small two-parameter eigenvalue problems that illustrate the theory from the previous sections.

EXAMPLE 4.1. If we take

$$W_1(\lambda, \mu) = (A_1 + \lambda B_1 + \mu C_1) = \begin{bmatrix} -\lambda - \mu & -1 \\ -1 & 1 \end{bmatrix},$$

$$W_2(\lambda, \mu) = (A_2 + \lambda B_2 + \mu C_2) = \begin{bmatrix} -2\lambda + \mu & -1 \\ -1 & 2 \end{bmatrix}$$

then

$$p_1(\lambda, \mu) = \det(W_1(\lambda, \mu)) = -\lambda - \mu + 1$$

$$p_2(\lambda, \mu) = \det(W_2(\lambda, \mu)) = -4\lambda + 2\mu + 1$$

and the problem is clearly regular. Its only eigenvalue is $(\lambda, \mu) = (\frac{1}{2}, \frac{1}{2})$. The corresponding operator determinants are:

$$\Delta_0 = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & -2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ have the same Kronecker structure which contains the following blocks: $L_0, 2N_1, L_0^T$, and $J_1(\frac{1}{2})$. The minimal reducing subspace is $\mathcal{R}(\Delta_1, \Delta_0) = \mathcal{R}(\Delta_2, \Delta_0) = \text{span}(e_4)$. The corresponding subspace for the two blocks N_1 is $\text{span}(e_2, e_3)$.

The assumptions of Theorems 3.5 and 3.7 are satisfied. A common regular right eigenvector for the eigenvalue $(\lambda, \mu) = (\frac{1}{2}, \frac{1}{2})$ is $[2 \ 1 \ 2 \ 1]^T + \alpha e_4$ for an arbitrary $\alpha \in \mathbb{C}$. If we take $\alpha = 0$, then $x = [2 \ 1 \ 2 \ 1]^T = [1 \ 1]^T \otimes [2 \ 1]^T$ is a decomposable regular eigenvector. As all matrices are symmetric and the eigenvalue is real, x is also a regular left eigenvector. One can see that $x^* \Delta_0 x = -6$ is nonzero as predicted by Corollary 3.3.

EXAMPLE 4.2. In [13], we show that one can solve a quadratic two-parameter eigenvalue problem by linearizing it as a singular two-parameter eigenvalue problem. In the Appendix of [13], we provide a linearization of an arbitrary polynomial two-parameter eigenvalue problem as a singular two-parameter eigenvalue problem. The new theory presented in this paper shows that all simple eigenvalues can be computed from the above linearization.

For an example we take the following system of two bivariate polynomials

$$(4.1) \quad \begin{aligned} p_1(\lambda, \mu) &= 1 + 2\lambda + 3\mu + 4\lambda^2 + 5\lambda\mu + 6\mu^2 + 7\lambda^3 + 8\lambda^2\mu + 9\lambda\mu^2 + 10\mu^3 = 0 \\ p_2(\lambda, \mu) &= 10 + 9\lambda + 8\mu + 7\lambda^2 + 6\lambda\mu + 5\mu^2 + 4\lambda^3 + 3\lambda^2\mu + 2\lambda\mu^2 + \mu^3 = 0. \end{aligned}$$

Following [13], we linearize the above system as a singular two-parameter eigenvalue problem, where

$$W_1(\lambda, \mu) = \begin{bmatrix} 1 & 2 & 3 & 4 + 7\lambda & 5 + 8\lambda & 6 + 9\lambda + 10\mu \\ \lambda & -1 & 0 & 0 & 0 & 0 \\ \mu & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \mu & 0 & 0 & -1 \end{bmatrix},$$

$$W_2(\lambda, \mu) = \begin{bmatrix} 10 & 9 & 8 & 7 + 4\lambda & 6 + 3\lambda & 5 + 2\lambda + \mu \\ \lambda & -1 & 0 & 0 & 0 & 0 \\ \mu & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & -1 & 0 \\ 0 & 0 & \mu & 0 & 0 & -1 \end{bmatrix}.$$

One can check that $\det(W_i(\lambda, \mu)) = p_i(\lambda, \mu)$ for $i = 1, 2$. The obtained two-parameter eigenvalue problem has 9 finite regular eigenvalues, which are all simple. The only real eigenvalue is $(\lambda, \mu) = (-2.4183, 1.8542)$ while the remaining 8 eigenvalues appear in conjugate pairs. All eigenvalues agree with the roots of the system (4.1).

In a similar way, an arbitrary system of two bivariate polynomials could be linearized as a singular two-parameter eigenvalue problem. This gives a new approach

for the numerical computation of roots of such systems. The dimension of the matrices of the linearized two-parameter eigenvalue problem is large, but they are also very sparse. Therefore, the new approach is most likely not competitive to advanced numerical methods that compute all solutions of polynomial systems, for instance, to the homotopy method PHCPack [17]. But, combined with a Jacobi-Davidson approach, it might be an alternative when one is interested only in part of the roots that are close to a given target. We plan to explore this in our further research.

EXAMPLE 4.3. For this example we take

$$W_1(\lambda, \mu) = \begin{bmatrix} 1 + \lambda + \mu & 0 & 0 \\ 0 & 1 + \lambda + \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W_2(\lambda, \mu) = \begin{bmatrix} 2 + 4\lambda + 6\mu & 0 \\ 0 & 1 \end{bmatrix}.$$

Now $p_1(\lambda, \mu) = (1 + \lambda + \mu)^2$, $p_2(\lambda, \mu) = 2 + 4\lambda + 6\mu$, and the problem has a double eigenvalue $(\lambda, \mu) = (-2, 1)$ with geometric multiplicity 2. Its right (and left) eigenvectors lie in the span of $e_1 \otimes e_1$ and $e_2 \otimes e_1$. Although the assumptions of Proposition 3.3 are not satisfied, we obtain $x^* \Delta_0 x = 2 \neq 0$ for $x = e_1 \otimes e_1$ and (3.9) is satisfied. It follows from Lemma 3.4 that $(-2, 1)$ is a regular finite eigenvalue of matrix pencils $\Delta_1 - \lambda \Delta_0$ and $\Delta_2 - \mu \Delta_0$.

EXAMPLE 4.4. We take

$$(4.2) \quad W_1(\lambda, \mu) = \begin{bmatrix} 2 + \lambda & 1 + 2\lambda & \lambda \\ \lambda & 2 + 2\lambda + 2\mu & \mu \\ \mu & 1 + 2\mu & 2 + \mu \end{bmatrix},$$

$$W_2(\lambda, \mu) = \begin{bmatrix} 1 + \lambda & 1 + 2\lambda & \lambda \\ \lambda & 1 + 2\lambda + 2\mu & \mu \\ \mu & 1 + 2\mu & 1 + \mu \end{bmatrix}.$$

Now

$$p_1(\lambda, \mu) = \lambda^2 + 6\mu\lambda + 10\lambda + \mu^2 + 10\mu + 8$$

$$p_2(\lambda, \mu) = (\lambda + \mu + 1)^2$$

and the problem has a quadruple eigenvalue $(\lambda, \mu) = (-\frac{1}{2}, -\frac{1}{2})$ that is geometrically simple.

The pencils $\Delta_1 - \lambda \Delta_0$ and $\Delta_2 - \mu \Delta_0$ have the same Kronecker structure with the following blocks: $L_0, 2N_2, L_0^T$, and $J_4(-\frac{1}{2})$. The minimal reducing subspaces are

$$\mathcal{R}(\Delta_1, \Delta_0) = \text{span} \left([0 \ 1 \ -2]^T \otimes [0 \ 1 \ -2]^T \right)$$

$$\mathcal{R}(\Delta_2, \Delta_0) = \text{span} \left([2 \ -1 \ 0]^T \otimes [2 \ -1 \ 0]^T \right).$$

A common regular eigenvector for pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ for the eigenvalue $(-\frac{1}{2}, -\frac{1}{2})$ is

$$z_1 = [0 \quad 1 \quad 0]^T \otimes [1 \quad -1 \quad 1]^T.$$

The corresponding common left eigenvector is $y = [1 \quad 2 \quad 1]^T \otimes [1 \quad 0 \quad 1]^T$. This gives $y^*\Delta_0 z_1 = 0$ and the condition (3.9) is not satisfied. The ascent of the eigenvalue $(-\frac{1}{2}, -\frac{1}{2})$ is 4. If we take vectors

$$\begin{aligned} z_2 &= [1 \quad 0 \quad -1]^T \otimes [1 \quad -1 \quad 1]^T, \\ z_3 &= [1 \quad 0 \quad 1]^T \otimes [1 \quad -1 \quad 1]^T + [0 \quad 1 \quad 0]^T \otimes [0 \quad 1 \quad 0]^T, \\ z_4 &= [1 \quad 0 \quad -1]^T \otimes [0 \quad 1 \quad 0]^T + [0 \quad 2 \quad 0]^T \otimes [1 \quad 0 \quad -1]^T \end{aligned}$$

and define subspaces $\mathcal{S}_i = \text{span}(z_1, \dots, z_i)$ for $i = 1, \dots, 4$, then $(\Delta_1 + \frac{1}{2}\Delta_0)\mathcal{S}_i \subset \Delta_0\mathcal{S}_{i-1}$ and $(\Delta_2 + \frac{1}{2}\Delta_0)\mathcal{S}_i \subset \Delta_0\mathcal{S}_{i-1}$ for $i = 2, 3, 4$. Vectors z_1, z_2, z_3 , and z_4 thus form a basis for the common root subspace of the eigenvalue $(-\frac{1}{2}, -\frac{1}{2})$.

Although the assumptions of Theorems 3.5 and 3.7 are not satisfied, we see that also in this case the finite regular eigenvalues of the two-parameter eigenvalue problem (4.2) agree with the finite regular eigenvalues of the associated system (1.4). We obtained the same for many other numerical examples with multiple eigenvalues and this indicates that the theory could probably be extended to cover a wider class of singular two-parameter eigenvalue problems.

5. Numerical methods. There are several numerical methods for two-parameter eigenvalue problems, see for instance [9] and references therein, but, most of the methods require that the problem is nonsingular. There are some exceptions, for instance, we can apply the Newton method [3] on (1.2), but this method requires a good initial approximation and computes only one eigenvalue. All methods that can compute all the eigenvalues require that the problem is nonsingular.

In [13], we present a numerical algorithm for the computation of the common regular eigenvalues of a pair of singular matrix pencils. The details can be found in [13]; let us just mention that the algorithm is based on the staircase algorithm for one matrix pencil from [16]. The algorithm returns matrices Q and U with unitary columns that define matrices $\tilde{\Delta}_i = Q^*\Delta_i U$ of size $k \times k$ for $i = 0, 1, 2$ such that $\tilde{\Delta}_0$ is nonsingular and the k common finite regular eigenvalues of (1.4) are the eigenvalues of the projected regular matrix pencils $\tilde{\Delta}_1 - \lambda\tilde{\Delta}_0$ and $\tilde{\Delta}_2 - \mu\tilde{\Delta}_0$. This algorithm can be applied to compute the eigenvalues of a general singular two-parameter eigenvalue problem.

When N is very large, it is not feasible anymore to apply the algorithm from [13] because of its complexity. For such problems, in particular when the matrices are

sparse, one could apply a Jacobi–Davidson method [9]. The only adjustment is that as it might happen that the smaller projected problem is singular, the routine for the solution of a smaller projected two-parameter eigenvalue problem should be replaced by a method that can handle singular problems.

6. Conclusion and acknowledgments. The results in this paper prove that for a large class of singular two-parameter eigenvalue problems (1.1) one could compute all eigenvalues by computing the common regular eigenvalues of the associated coupled pair of generalized eigenvalue problems (1.4). The theory guarantees that this works for all algebraically simple eigenvalues. Various numerical results suggest that this approach is correct for all finite regular eigenvalues of a singular two-parameter eigenvalue problem.

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