On the composition conjecture for a class of rigid systems

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Abstract. In this paper, we prove that for a class of rigid systems the Composition Conjecture is correct. We show that the Moments Condition is the sufficient and necessary conditions for these rigid systems to have a center at origin point. By the obtained conclusions we can derive all the focal values of these higher order polynomial differential systems and their expressions are more succinct and beautiful.

Keywords: Composition Conjecture, rigid system, Center Condition, Moments Condition.

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1 Introduction

Consider the planar differential system

\[
\begin{cases}
x' = -y + p(x, y), \\
y' = x + q(x, y)
\end{cases}
\] (1.1)

where \(p, q\) are analytic functions starting with second order terms.

If \(p\) and \(q\) are polynomials of degree \(n\) and \(yp - xq = 0\), the system (1.1) in polar coordinates becomes

\[
\frac{dr}{d\theta} = r^2 \left( \sum_{i=0}^{n-2} A_i(\theta) r^i \right),
\] (1.2)

where \(A_i(\theta) (i = 0, 1, 2, \ldots, n - 2)\) are \(2\pi\)-periodic functions. Therefore, the system (1.1) has a center at \((0, 0)\) if and only if all solutions \(r(\theta)\) of equation (1.2) near the solution \(r = 0\) are periodic. In such case it is said that equation (1.2) has a center at \(r = 0\) \([15,20]\).

If \(p\) and \(q\) are homogeneous polynomials of degree \(n\), via the Cherkas \([9]\) transformation equation (1.2) becomes the Abel equation

\[
\frac{dp}{d\theta} = p^2 (\tilde{A}_1(\theta) + \tilde{A}_2(\theta) p),
\] (1.3)

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where $\tilde{A}_i(\theta)$ ($i = 1, 2$) are $2\pi$-periodic functions. Thus, finding the center conditions for (1.1) is equivalent to studying when the Abel equation (1.3) has a center at $\rho = 0$. This problem has been investigated in [5, 8, 10, 14, 20] among other works.

The problem of determining necessary and sufficient conditions on $p$ and $q$ for system (1.1) to have a center at the origin is known as the center-focus problem. Due to the Hilbert’ basis theorem we know that when $p$ and $q$ are polynomials of a given degree this set of conditions is finite. To get the necessary and sufficient conditions is very complexity, up to now only for a very few families of polynomial system (1.1) the center conditions are known. The problem is solved for quadratic system and some families of cubic systems and systems in the form of the linear center perturbed by homogeneous quartic and quintic nonlinearities, see e.g. [1, 4, 9–16, 22] and references given there.

Alwash and Lloyd [6,7] give the following simple sufficient condition for the Abel equation to have a center.

**Theorem 1.1** ([6,7]). If there exists a differentiable function $u$ of period $2\pi$ such that

$$\tilde{A}_1(\theta) = u'(\theta) \tilde{A}_1(u(\theta)), \quad \tilde{A}_2(\theta) = u'(\theta) \tilde{A}_2(u(\theta))$$

for some continuous functions $\tilde{A}_1$ and $\tilde{A}_2$, then the Abel differential equation (1.3) has a center at the origin.

The following statement presents a generalization of Theorem 1.1.

**Theorem 1.2** ([2,22]). If there exists a differentiable function $u$ of period $2\pi$ such that

$$A_i(\theta) = u' \hat{A}_i(u), \quad (i = 1, 2, \ldots, n - 2)$$

for some continuous functions $\hat{A}_i$ ($i = 1, 2, \ldots, n - 2$), then the differential equation (1.2) has a center at $r = 0$.

The condition in Theorem 1.1 (or Theorem 1.2) is called the Composition Condition. When an Abel equation (or (1.2) ) has a center because its coefficients satisfy the composition condition we will say that this equation has a CC-center. In [7, 18] it was shown that this condition is not necessary to have a center.

The Composition Conjecture is that the composition condition in Theorem 1.1 (or Theorem 1.2) is not only the sufficient but also necessary condition for a center. This conjecture first appeared in [6] with classes of coefficients which are polynomial functions in $t$, or trigonometric polynomials. A counterexample was presented in [7, 13] to demonstrate that the conjecture is not true. To find the restrictive conditions under which the composition conjecture is true, this is an open problem which has attracted during the last years a wide interest [2–8,10,12,14,17,18,21,22]. The authors in paper [11] give the sufficient and necessary conditions for the $r = 0$ of the Abel equation (1.3) to be a CC-center.

The condition

$$\int_0^{2\pi} \left( \int_0^\theta \tilde{A}_1(\tau) d\tau \right)^k \tilde{A}_2(\theta) d\theta = 0 \quad (k \geq 0)$$

is called Moments Condition [10]. In [18] an example of a polynomial Abel equation satisfying the moments condition and not satisfying the composition condition is given. Later on, in [17] a full algebraic characterization of the moments condition in the polynomial case is done. In [10] prove that a natural trigonometric analogous to it does not hold.
In [21], it was proved that for the rigid system
\[
\begin{align*}
\dot{x}' &= -y + xP, \\
\dot{y}' &= x + yP
\end{align*}
\] (1.4)
with \(P = P_1 + P_m\), \(P_m\) is a homogeneous polynomial of degree \(m\) and which is an arbitrary natural number greater than 1, the composition conjecture is true, i.e., its origin point is a center and a CC-center, and shown that for its corresponding \(2\pi\)-periodic equation
\[
\frac{dr}{d\theta} = r(P_1(\cos \theta, \sin \theta)r + P_m(\cos \theta, \sin \theta)r^m),
\]
r = 0 is a center if and only if it satisfies the moments conditions:
\[
\int_0^{2\pi} \left( \int_0^\theta P_1(\cos \tau, \sin \tau)d\tau \right)^k P_m(\cos \theta, \sin \theta)d\theta = 0, \quad (k = 0, 1, 2, \ldots, m).
\]

In [1] the authors used the methods of the normal form theory to prove that the rigid system (1.4) has a center if and only if it is reversible. In [19], the author has calculated by computer and obtained the center condition for system (1.4) with \(P = P_m + P_{2m}\), \(m\) is a finite number that does not exceed 5.

In this paper, we will study the rigid system
\[
\begin{align*}
\dot{x}' &= -y + x(P_1(x, y) + P_m(x, y) + P_{2m+1}(x, y)), \\
\dot{y}' &= x + y(P_1(x, y) + P_m(x, y) + P_{2m+1}(x, y)),
\end{align*}
\] (1.5)
where \(P_k(x, y)\) is a homogeneous polynomial in \(x, y\) of degree \(k\) and \(P_1 \neq 0\), \(m\) is an arbitrary positive integer greater than 1, and give the necessary and sufficient conditions for the origin of (1.5) to be a center. We prove that under some restrictions conditions the composition conjecture is true for its corresponding periodic differential equation
\[
\frac{dr}{d\theta} = r(P_1(\cos \theta, \sin \theta)r + P_m(\cos \theta, \sin \theta)r^m + P_{2m+1}(\cos \theta, \sin \theta)r^{2m+1}).
\] (1.6)

By this, we can derive all the focal values of system (1.5) and they contain exactly \(\left\lfloor \frac{m^2}{2} \right\rfloor + m + 2\) relations. As \(m\) is an arbitrary number, in general, even with the help of computers, it is difficult to get the center conditions. However, in this paper, we have obtained these results only by using the mathematical analysis technique. We firmly believe that the method of this paper can be used to solve the center-focus problem of more high-order polynomial differential systems.

In the following we denote \(\bar{P} = \int_0^\theta P(\cos \theta, \sin \theta)d\theta; C_n^k = \frac{n!}{k!(n-k)!}(0 \leq k \leq n); C_n^k = 0, \) if \(k < 0\) or \(n < 0; \sum_{i+j=k} = 0, \) if \(k < 0.\)

## 2 Several lemmas

In order to prove the main result, first of all, we give the following lemmas.

**Lemma 2.1** ([21]). If \(P_1 \neq 0\) and for an arbitrary positive integer \(m,\)
\[
\int_0^{2\pi} \bar{P}_1(\cos \theta, \sin \theta)P_m(\cos \theta, \sin \theta)d\theta = 0 \quad (i = 0, 1, 2, \ldots, m),
\]
then
\[ P_m = P_1 \sum_{j=1}^{m} j \lambda_j \vec{p}_1^{j-1}, \] (2.1)

where \( \lambda_j \) (j = 1, 2, ..., m) are real numbers.

**Lemma 2.2.** Suppose that \( P_m \) satisfies (2.1) and
\[ h'_0 = P_m, \quad h'_k = 2P_1 \sum_{i+j=k-1} \vec{p}_1^i h_j + P_m C_{m+k}^m \vec{p}_1^k, \quad (k = 1, 2, 3, \ldots) \] (2.2)

Then
\[ h_k = \sum_{j=0}^{k} h'_k \sum_{j} \vec{p}_1^{k-j} \vec{p}_1^j P_m = \hat{h}_k \hat{p}_1^{m+k} + \sum_{j=0}^{k-1} \hat{h}_{kj} \hat{p}_1^{m+j}, \] (2.3)

where \( \hat{h}_{kj} \) (j = 0, 1, 2, ..., k − 1) are real numbers,
\[ h'_j = \frac{2}{k-j} \sum_{i=j}^{k-1} h'_i = (k-j+1) C_{m+j-2}^j, \quad (j = 0, 1, 2, \ldots, k-1), \] (2.4)
\[ \hat{h}_k = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} h'_k = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} (k-j+1) C_{m+j-2}^j. \] (2.6)

**Proof.** By (2.1) we get
\[ \vec{p}_m = \int_0^\theta P_m d\theta = \lambda_m \vec{p}_1^m + \sum_{j=1}^{m-1} \lambda_j \vec{p}_1^j, \] (2.7)
\[ \vec{p}_1 \vec{p}_m = \int_0^\theta P_1 \sum_{j=1}^{m} j \lambda_j \vec{p}_1^{j+k} = \frac{m}{k+m} \lambda_m \vec{p}_1^{m+k} + \sum_{j=1}^{m-1} \frac{j}{j+k} \lambda_j \vec{p}_1^{j+k}, \] (2.8)

where \( \lambda_j \) (j = 1, 2, ..., m) are real numbers.

By (2.2) and (2.7) and (2.8) we get \( h'_0 = P_m, \)
\[ h'_0 = \hat{P}_m = h'_0 \hat{p}_1^m + \sum_{j=1}^{m-1} \lambda_j \hat{p}_1^j, \quad h'_0 = 1, \quad \hat{h}_0 = \lambda_m h'_0 = \lambda_m. \]
\[ h'_1 = 2P_1 h'_0 + P_m C_{m+1}^m \hat{p}_1, \]
\[ h_1 = 2h'_0 P_1 + (C_{m+1}^1 - 2h'_0) \hat{p}_1 P_m = h'_0 P_1 + h'_1 \hat{p}_1 P_m = \hat{h}_1 \hat{p}_1^{m+1} + \sum_{j=1}^{m} \hat{h}_{1j} \hat{p}_1^j, \]

where \( \hat{h}_{1j} \) (j = 1, 2, ..., m) are real numbers,
\[ h'_1 = 2h'_0, \quad h'_1 = C_{m+1}^1 - h'_1 = C_{m-1}^1, \quad \hat{h}_1 = \left( h'_0 + h'_1 \frac{m}{1+m} \right) \lambda_m = \frac{m^2 + m + 2}{m+1} \lambda_m, \]

this means that the relations (2.3)–(2.6) are valid for \( k = 0, 1 \). Now suppose that these identifies are valid for integer \( k \), next we will prove they are correct for integer \( k + 1 \).
Indeed, by (2.2)–(2.5) and (2.7) and (2.8) we have

\[ h_{k+1}' = 2P_1 \sum_{i+j=k} \bar{P}_i^j h_j + P_m C_{m+k+1}^m \bar{P}_1^k, \]

\[ = 2P_1 \sum_{j=0}^{k} \bar{P}_1^{k-j} \bar{P}_1^j P_m \sum_{i=j}^{k} h_i^j + P_m C_{m+k+1}^m \bar{P}_1^k, \]

solving this equation we get

\[ h_{k+1} = \sum_{j=0}^{k+1} h_{k+1}' h_{k+1}^j \bar{P}_1^j P_m = \hat{h}_{k+1} + \sum_{j=1}^{m+k} \hat{h}_{k+1}^j / \bar{P}_1^j, \]

where \( \hat{h}_{k+1}^j \) \( (j = 1, 2, \ldots, m + k) \) are real numbers,

\[ \hat{h}_{k+1} = \lambda_m \sum_{j=0}^{k+1} \frac{m^j}{m+j} h_{k+1}^j, \]

\[ h_{k+1}' = \frac{2}{k+1} \sum_{j=0}^{k} h_{k+1}^j, \]

(2.9)

By mathematical induction, the present lemma holds. \( \square \)

Similarly, we can prove the following lemma.

**Lemma 2.3.** If

\[ \delta_k' = 2P_1 \sum_{i+j=k-1} \bar{P}_i^j \delta_j + P_{2m+1} C_{2m+k+1}^{2m+1} \bar{P}_1^k, \]

then

\[ \delta_k = \sum_{j=0}^{k} \delta_k^j \bar{P}_1^{k-j} P_{2m+1}, \]

where

\[ \delta_k^j = \frac{2}{k-j} \sum_{i=j}^{k} \delta_i^j = (k-j+1) C_{2m+j-1}^j, \]

\( (j = 0, 1, 2, \ldots, k-1), \)

\[ \delta_k = C_{2m+k+1}^{2m+1} - \sum_{j=0}^{k-1} \delta_k^j = C_{2m+k-1}^k, \]

\( \delta_0^0 = 1. \)

**Lemma 2.4.** If the conditions of Lemma 2.2 are satisfied and

\[ \alpha_0' = C_{m+1}^{m+1} h_0, \]

\[ \alpha_k' = P_1 \sum_{i+j=k-1} (h_i h_j + 2 \bar{P}_i^j \alpha_j) + 2P_1 \sum_{i+j=k-1} h_i \alpha_j + P_m C_{m+1}^m \sum_{i+j=k-1} \bar{P}_1^i h_j + P_m C_{m+1}^m \sum_{i+j=k-1} \bar{P}_1^i \alpha_j, \]

(2.11)
then
\[ \alpha_k = \hat{\alpha}_k P_{1}^{2m+k} + \sum_{j=2}^{2m+k-1} \hat{\alpha}_k \hat{P}_j^1, \quad (k = 0, 1, 2, \ldots), \] (2.12)
where \( \hat{\alpha}_k, (j = 2, 3, \ldots, 2m + k - 1) \) are real numbers and
\[
\hat{\alpha}_0 = \frac{m+1}{2} \lambda_{m}^2 \\
\hat{\alpha}_k = \frac{1}{2m+k} \left( \sum_{i+j=k-1} \hat{h}_i \hat{h}_j + 2 \sum_{j=0}^{k-1} \hat{\alpha}_j + 2 \sum_{i+j=k-1-m} \hat{h}_i \hat{\alpha}_j + m \lambda_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^m \hat{h}_j + \lambda_m m \sum_{i+j=k-m} C_{m+i-1}^m \hat{h}_j \right) + \lambda_m m \sum_{i+j=k-m} C_{m+i-1}^m \hat{h}_j, \quad (k = 1, 2, \ldots). 
\] (2.13)

Proof. Solving equation \( \alpha'_0 = C_{m+1}^1 P_m h_0 \), by Lemma 2.2 we get
\[
\alpha_0 = \frac{1}{2} C_{m+1}^1 h_0^0 \alpha_0^2 + \sum_{j=2}^{2m+k-1} \hat{\alpha}_0 \hat{P}_j^1 
\]
where \( \hat{\alpha}_0 \) are real numbers, \( \hat{\alpha}_0 = \frac{1}{2} C_{m+1}^1 \lambda_{m}^2 \). Thus, the relation (2.12) is valid for \( k = 0 \).

Suppose that the conclusion of the present lemma is correct for integer \( k - 1 \), next we will show that they are held for integer \( k \).

Indeed, by Lemma 2.2 and (2.11) we get
\[
\alpha'_k = P_{1}^{2m+k-1} \left( \sum_{i+j=k-1} \hat{h}_i \hat{h}_j + 2 \sum_{j=0}^{k-1} \hat{\alpha}_j + 2 \sum_{i+j=k-1} \hat{h}_i \hat{\alpha}_j + m \lambda_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^m \hat{h}_j + \lambda_m m \sum_{i+j=k-m} C_{m+i-1}^m \hat{h}_j \right) + \ldots 
\]
which implies that the identities (2.12) and (2.13) are valid. By mathematical induction, the present lemma holds.

\[ \square \]

Lemma 2.5. Suppose that \( h_k \) and \( \delta_k \) are the solutions of equations (2.2) and (2.9) respectively, and
\[
\beta'_k = 2P_{1} \sum_{i+j=k-1} (P_{1} \beta_j + h_i \delta_j) + P_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^m \hat{P}_j^1 \delta_j + P_{2m+1} C_{2m+2}^1 \sum_{i+j=k} C_{2m+2}^1 \hat{P}_j^1 h_j, \quad (k = 0, 1, 2, \ldots, m+1) 
\] (2.14)

Then
\[
\beta_k = \sum_{j=0}^{k} (\beta'_k P_{1}^{m+k-j} P_{2m+1} + \beta_j P_{1}^{m+k-j} P_{2m+1} + \sum_{j=0}^{m+k-1} \beta_{k_j} P_{1}^{j} P_{2m+1}), 
\] (2.15)

where \( \beta_{k_j}, (i = 0, 1, 2, \ldots, j) \) are real numbers,
\[
\beta'_k = C_{2m+j-1}^1 \hat{h}_{k-j}, \quad (j = 0, 1, 2, \ldots, k), 
\] (2.16)
\[
\beta_{m+j}^m = (k-j+1) \beta_{m+j}^1, \quad \beta_{0}^m = C_{m+1}^1 \hat{h}_0, 
\] (2.17)
\[
\beta_{m+k}^m = \sum_{j=0}^{k} (C_{2m+j-2}^j C_{2m+j-1}^1 - C_{2m+j-1}^1 C_{m+k-j-1}^1) \hat{h}_{k-j} - \sum_{j=0}^{k} (k-j+1) \beta_{m+j}^m. 
\] (2.18)
Proof. By (2.14) we get

\[ \beta_0^* = C_{m+1}^1 P_m \delta_0 + P_{2m+1} C_{2m+2}^l h_0, \]

using Lemma 2.2 and Lemma 2.3 we get

\[ h_0 = h_0^0 P_m, \quad \delta_0 = \delta_0^0 P_{2m+1}, h_0^0 = \delta_0^0 = 1, \]

thus

\[ \beta_0^* = C_{m+1}^1 P_m \delta_0^0 P_{2m+1} + P_{2m+1} C_{2m+2}^l h_0^0 P_m, \]

\[ \beta_0 = C_{m+1}^1 \delta_0^0 P_m P_{2m+1} + (C_{2m+2}^l h_0^0 - C_{m+1}^l h_0^0) P_m P_{2m+1} \]

\[ = \beta_0^1 P_{2m+1} + \beta_0^m P_{2m+1} + \sum_{j=0}^{m-1} (\beta_{0j} P_{2m+1}^j + \beta_{0j} P_{2m+1}^{m+j}), \]

where \( \beta_{0j} \) and \( \beta_{00j} \) are real numbers,

\[ \rho_0^m = \lambda_m C_{m+1}^l \delta_0^0 = \lambda_m C_{m+1}^{l'}, \quad \rho_0^m = \lambda_m (C_{2m+2}^l h_0^0 - C_{m+1}^l h_0^0) = \lambda_m C_{m+1}^{l'}, \]

Therefore, the conclusion of the present lemma is correct for \( k = 0 \). Now we suppose that

\[ \beta_n = \sum_{j=0}^{n} (\beta_{n+j} P_{2m+1}^{n+j} + \rho_{n+j} P_{2m+1}^{n+j}) \quad (n = 0, 1, 2, \ldots, k - 1). \]

By this and (2.14) we get

\[ \beta_k^* = 2P_1 \sum_{i+j=k+1} \sum_{j=0}^{i+j} (\beta_{i+j} P_{2m+1}^{i+j} + \rho_{i+j} P_{2m+1}^{i+j}) \]

\[ + 2P_1 \sum_{i+j=k} \sum_{i=0}^{j} \delta_{i} P_{2m+1}^{i+j} + m \lambda_m C_{m+1}^{l'} \sum_{i+j=k} \sum_{i=0}^{j} \delta_{i} P_{2m+1}^{i+j} \]

\[ + P_{2m+1} P_{2m+2} \sum_{i+j=k} C_{2m+2}^{l'} \hat{h}_j + \cdots, \]

solving this equation we get

\[ \beta_k = \sum_{j=0}^{k} (\beta_{k+j} P_{2m+1}^{k+j} + \rho_{k+j} P_{2m+1}^{k+j}) + \sum_{j=0}^{m-k} \sum_{i=0}^{j} (\beta_{k+j} P_{2m+1}^{i+j}), \]

where \( \beta_{k+j} \) are real numbers,

\[ \beta_k^1 = \frac{1}{m+k-j} \left( 2 \sum_{i=0}^{k-1} \delta_{k-i} + 2 \sum_{i=0}^{k-1} \beta_{i} + m \lambda_m C_{m+1}^{l'} C_{2m+2}^{l'} \right), \quad (j = 0, 1, 2, \ldots, k - 1), \]

\[ \beta_k^m = \lambda_m C_{m+1}^{l'} C_{2m+k-1}^{l'}, \quad (j = 0, 1, 2, \ldots, k - 1), \]

\[ \rho_k^{m+j} = \frac{2}{k-j} \sum_{i=0}^{k-j} \beta_{i}^{m+j}, \quad (j = 0, 1, 2, \ldots, k - 1), \]

\[ \rho_k^{m+k} = C_{2m+2}^{l'} \sum_{i+j=k} C_{2m+2}^{l'} \hat{h}_j - \beta_k^1 - \sum_{j=0}^{k-1} (\beta_k^1 + \beta_k^{m+j}). \]
In order to prove the relations (2.16)–(2.18) are valid, first of all, we use mathematical induction to show that

$$2 \sum_{j=0}^{k} \hat{h}_j + \lambda_m m C_{m+k+1}^m = (m + k + 1) \hat{h}_{k+1} \quad (k = 0, 1, 2, \ldots). \tag{2.23}$$

Indeed, by (2.6) we get $\hat{h}_0 = \lambda_m$, $\hat{h}_1 = \lambda_m (2 + \frac{m}{m+1} C_{m-1}^1)$, thus

$$2 \hat{h}_0 + \lambda m m C_{m+1}^m = \lambda m (2 + m C_{m+1}^1) = (m + 1) \hat{h}_1,$$

this means that the relation (2.23) is valid for $k = 0$. Suppose that the identity (2.23) is valid for positive integer $k$, next we will show that

$$2 \sum_{j=0}^{k+1} \hat{h}_j + \lambda m m C_{m+k+2}^m = (m + k + 2) \hat{h}_{k+2}. \tag{2.24}$$

Indeed, from the inductive hypothesis and (2.6) we have

$$2 \sum_{j=0}^{k+1} \hat{h}_j = (m + k + 1) \hat{h}_{k+1} - \lambda m m C_{m+k+1}^m + 2 \hat{h}_{k+1}$$

$$= \lambda m (m + k + 3) \sum_{j=0}^{k+1} \frac{m}{m+j} (k + 2 - j) C_{m+j-2}^j - \lambda m m C_{m+k+1}^m.$$

By this and $C_j^i = C_{i-1}^i + C_{i-1}^{i-1}$, we get

$$(m + k + 2) \hat{h}_{k+2} - 2 \sum_{j=0}^{k+1} \hat{h}_j = (m + k + 2) \lambda m \sum_{j=0}^{k+1} \frac{m}{m+j} (k + 3 - j) C_{m+j-2}^j$$

$$- \lambda m (m + k + 3) \sum_{j=0}^{k+1} \frac{m}{m+j} (k + 2 - j) C_{m+j-2}^j + \lambda m m C_{m+k+1}^m$$

$$= \lambda m m \sum_{j=0}^{k+1} C_{m+j-2}^m + \lambda m m C_{m+k+2}^m$$

$$= \lambda m m \sum_{j=0}^{k+2} C_{m+j-2}^j + \lambda m m C_{m+k+2}^m = m \lambda m C_{m+k+2}^m.$$

Thus the identity (2.24) is correct. By mathematical induction, the identity (2.23) is valid.

Now, we prove that the relation (2.16) is correct. Indeed, by (2.19) and (2.20) we get

$$\beta_0^0 = \lambda m C_{m+1}^1 = C_{m+1}^1 \hat{h}_0,$$

and

$$\beta_0^1 = \frac{1}{m+1} (2 \hat{h}_0 + 2 \beta_0 + \lambda m m C_{m+1}^1 C_{m+2}^1)$$

$$= \frac{1}{m+1} C_{m+2}^1 (2 + m C_{m+1}) = C_{m+2}^1 \hat{h}_1,$$

and

$$\beta_1^1 = \lambda m C_{2m}^1 C_{m+1}^1 = C_{2m}^1 C_{m+1}^1 \hat{h}_0.$$
Thus, the identity (2.16) is correct for \( k = 0,1 \). Suppose that (2.16) is valid for natural numbers less than or equal to \( k \), next we will prove that (2.16) holds for \( k + 1 \).

In fact, by (2.19) and (2.23) and Lemma 2.4 we get

\[
\beta_{k+1}^j = \frac{1}{m+k+1-j} \left( 2 \sum_{i=0}^{k-j} \hat{h}_i \delta_{k-i}^j + 2 \sum_{i=j}^{k} \beta_i^j + m \lambda \mu C_{m+1}^1 C_{2m+j-1}^j C_{k+j+1}^{k-j+1} \right) \\
= \frac{1}{m+k+1-j} C_{2m+j-1}^j C_{k+m+1}^1 \left( 2 \sum_{i=0}^{k-j} \hat{h}_i + \lambda_m m C_{m+k+1}^m \right) = C_{2m+j-1}^j C_{k+m+1}^1 \beta_{k+1}^j .
\]

So, by mathematical induction the identity (2.16) is correct.

Now we prove that the identity (2.17) is correct.

As \( \beta_0^m = C_{m+1}^1 \lambda_m \), by (2.21) we get \( \beta_1^m = 2 \beta_0^m \), thus, the relation (2.17) is correct for \( k = 1 \).

Suppose that

\[ \beta_{k-1}^{m+j} = (k-j) \beta_j^m, \quad (j = 0,1,2,\ldots,k-2), \]

by this and (2.21) we get

\[ \beta_k^{m+j} = \frac{2}{k-j} \sum_{i=j}^{k-1} \beta_i^{m+j} = \frac{2}{k-j} \sum_{i=j}^{k-1} (i-j+1) \beta_{i}^{m+j} = (k-j+1) \beta_{j}^{m+j}. \]

Therefore, the identity (2.17) is valid.

Substituting (2.16) and (2.17) into (2.22) implies that the identity (2.18) holds.

In summary, the conclusions of the present lemma is correct. \( \square \)

**Lemma 2.6.** If \( \lambda_m \geq 0 \) and \( m \geq 2 \), then

\[ \beta_k^{m+k} \geq 0, \quad (k = 0,1,2,\ldots,m+1). \]  

**Proof.** As \( \lambda_m \geq 0 \), by (2.6) we get

\[ \hat{h}_k = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} (k-j+1) C_{m+j-2}^j \geq 0, \quad (k = 0,1,2,\ldots,m+1). \]

Using (2.17) we get

\[ \beta_0^m = C_{m+1}^1 \hat{h}_0 = \lambda_m C_{m+1}^1 \geq 0. \]

By this and using (2.18) we have

\[ \beta_1^{1+m} = C_{m+1}^1 \hat{h}_1 + C_{m+1}^1 C_{2m+2}^1 \hat{h}_0 - 2 \beta_0^m = C_{m+1}^1 \hat{h}_1 + C_{m+1}^1 C_{2m+2}^1 \hat{h}_0 \geq 0. \]

Thus the inequality (2.25) is valid for \( k = 0 \) and \( k = 1 \). Now we suppose that inequality (2.25) holds for natural numbers less than \( k \), i.e., \( \beta_k^{1+m} \geq 0, \quad (i = 0,1,2,\ldots,k-1) \). Next we will show that \( \beta_k^{k+m} \geq 0 \).

Indeed, by (2.18) we get

\[
\beta_{k-1}^{k+m} \geq \sum_{j=0}^{k-1} \left( C_{2m+2}^j C_{2m+j-1}^{j-1} + C_{2m+j-1}^j C_{m+1}^1 C_{m+k-j-k}^{k-j-1} \right) \hat{h}_{k-1-j} - \sum_{j=0}^{k-2} (k-j) \beta_j^{m+j} \\
= \sum_{j=1}^{k} \left( C_{2m+2}^j C_{2m+j-2}^{j-2} + C_{2m+j-2}^j C_{m+1}^1 C_{m+k-j-k}^{k-j-1} \right) \hat{h}_{k-j} - \sum_{j=0}^{k-2} (k-j) \beta_j^{m+j} .
\]
By this and (2.18) we get

\[ \beta_{k}^{m} = C_{m+1-k}^{1} \hat{h}_{k} + \sum_{j=1}^{k} (C_{2m+2}^{1} C_{2m+1}^{j-1} + C_{2m+1}^{1} C_{m+1+j-k}^{1}) \hat{h}_{k-j} - \sum_{j=0}^{k-2} (k-j) \beta_{j}^{m} - 2 \beta_{k-1}^{m} \]

\[ = \sum_{j=0}^{k-2} (k-j) \beta_{j}^{m+1} + C_{m+1-k}^{1} \hat{h}_{k} + (C_{2m+2}^{1} + C_{m+2-k}^{1} C_{2m+2}^{1}) \hat{h}_{k-1} \]

\[ + \sum_{j=2}^{k} \left( \frac{1}{j-1} C_{2m+2}^{1} C_{2m+j-2}^{j-2} C_{2m+j-1}^{1} + \frac{1}{j} C_{m+1+j-k}^{1} C_{2m+j-2}^{1} C_{2m-j}^{1} \right) \hat{h}_{k-j} \]

(2.26)

as \( \beta_{i}^{i+m} \geq 0 \) (\( i = 0, 1, 2, \ldots, k-2 \)) and \( \hat{h}_{i} > 0 \) (\( i = 0, 1, 2, \ldots, m+1 \)) and \( m \geq 2 \), then \( \beta_{k}^{k+m} \geq 0 \) holds. Thus, by mathematical induction, the present lemma is valid. The proof is complete.

**Lemma 2.7.** Suppose that \( h_{k} \) and \( \delta_{k} \) are the solutions of equations (2.2) and (2.9) respectively, \( \alpha_{k} \) and \( \beta_{k} \) are the solutions of equations (2.11) and (2.14) respectively and

\[ \hat{p}_{1}^{1} = C_{m+1}^{1} P_{m} \beta_{0} + P_{m} C_{m+1}^{1} h_{0} \delta_{0} + P_{2m+1} (C_{2m+2}^{1} h_{2}^{2} + C_{2m+2}^{1} \alpha_{0}), \]

(2.27)

\[ \hat{p}_{2}^{2} = 2 P_{1} (h_{0} \beta_{0} + \alpha_{0} \delta_{0} + \hat{p}_{1}^{1}) + P_{m} C_{m+1}^{1} \left( \sum_{i+j=1} C_{m-i+1}^{m-1} \hat{p}_{i}^{1} \beta_{j}^{1} + C_{m}^{1} \sum_{i+j+1=1} C_{m-i+2}^{m-2} \hat{p}_{i}^{1} h_{j} \delta_{1} \right) \]

\[ + P_{2m+1} \left( \sum_{i+j+1=1} C_{2m+2}^{2} C_{2m+1}^{j-i} \hat{p}_{i}^{1} h_{j} + C_{2m+2}^{1} \sum_{i+j=1} C_{2m+1}^{m-i+1} \hat{p}_{i}^{1} \alpha_{j} + C_{2m+2}^{1} \delta_{0} \right). \]

(2.28)

Then

\[ \hat{p}_{1} = \bar{p}_{1}^{0} \hat{p}_{2}^{1} \bar{p}_{2}^{2} \bar{p}_{2m+1} + \tilde{p}_{1}^{m} \hat{p}_{m}^{n} \bar{p}_{m}^{n} \bar{p}_{2m+1} + \bar{p}_{1}^{2m} \bar{p}_{2m}^{2m} \bar{p}_{2m+1} + \cdots, \]

(2.29)

\[ \hat{p}_{2} = \bar{p}_{2}^{0} \bar{p}_{1}^{0} \bar{p}_{2}^{0} \bar{p}_{2m+1} + \bar{p}_{2}^{1} \bar{p}_{1}^{1} \bar{p}_{2}^{1} \bar{p}_{2m+1} + \bar{p}_{2}^{2} \bar{p}_{1}^{2} \bar{p}_{2}^{2} \bar{p}_{2m+1} + \bar{p}_{2}^{2m} \bar{p}_{1}^{2m} \bar{p}_{2}^{2m} \bar{p}_{2m+1} + \bar{p}_{1}^{2m} \bar{p}_{2}^{2m} \bar{p}_{2m+1} + \bar{p}_{1}^{2m} \bar{p}_{2}^{2m} \bar{p}_{2m+1} + \cdots, \]

(2.30)

where

\[ \bar{p}_{1}^{0} = \frac{\lambda_{2}^{2}}{2} C_{m+1}^{1} C_{2m+1}^{1}, \quad \bar{p}_{1}^{m} = \lambda_{2}^{2} (C_{m+1}^{1})^{2}, \quad \bar{p}_{1}^{2m} = \lambda_{2}^{2} C_{m+1}^{1} C_{2m+1}^{1}, \]

\[ \bar{p}_{2}^{0} = \frac{1}{2m+1} \lambda_{m}^{2} C_{m+1}^{1} (4m^{3} + 4m^{2} + 6m + 4), \quad \bar{p}_{2}^{1} = \lambda_{m}^{2} m C_{m+1}^{1} C_{2m+1}^{1}, \]

\[ \bar{p}_{2}^{2m} = \lambda_{m}^{2} m (3m^{2} + 5m + 4), \quad \bar{p}_{2}^{2m} = \lambda_{m}^{2} (m^{3} + 3m^{2} + 4m + 4), \quad \bar{p}_{2}^{2m} = 2 \bar{p}_{1}^{2m} \]

\[ \bar{p}_{2m+1}^{2m} = 4 \lambda_{m}^{2} (m^{2} + m + 1). \]

**Proof.** By Lemmas 2.2-2.5, and substituting (2.3) and (2.10) and (2.12) and (2.15) into the equations (2.27) and (2.28) and solving them we get (2.29) and (2.30) and in which

\[ \bar{p}_{1}^{1} = \frac{\lambda_{m}^{2}}{2} C_{m+1}^{1} (\beta_{0}^{0} + m \lambda_{m}), \quad \bar{p}_{1}^{m} = \lambda_{m} C_{m+1}^{1} \beta_{0}^{m}, \]

\[ \bar{p}_{1}^{2m} = \lambda_{m}^{2} (m + 1)(3m + 2) - \frac{\lambda_{m}^{2}}{2} C_{m+1}^{1} (\beta_{0}^{0} + m \lambda_{m} + 2 \beta_{0}^{m}). \]

\[ \bar{p}_{2}^{0} = \frac{1}{2m+1} (2 \delta_{0}^{0} \beta_{0}^{0} + 2 \delta_{0}^{0} + 2 \lambda_{m} C_{m+1}^{1} (\beta_{0}^{0} + m \beta_{0}^{0}) + \lambda_{m} m C_{m+1}^{1} (\delta_{0}^{0} + \delta_{0}^{0} + \delta_{m-1}^{0})). \]
Moreover, this center is a CC-center. Where the expressions of Theorem 3.1.

Suppose that $m$ the composition conjecture is correct. In the following we only consider integer case of $m$ where $c$

Equating the corresponding coefficients of $a'$ where $\delta$

$\tilde{\beta}_2^{m+1} = \lambda_m C_{m+1}^{1} \tilde{\beta}_1^{m+1}$,

$\tilde{\beta}_2^{m} = \frac{\lambda_m}{m+1} (2\beta_0^m + C_m C_{m+1} \beta_1^m + C_{m+1} m^2 \beta_0^m + 2 \tilde{\beta}_1^m)$, $\tilde{\beta}_2^m = 2 \tilde{\beta}_1^m$,

$\tilde{\beta}_2^{2m+1} = C_{2m+2}^{1} (2 \theta h_1 + C_{2m} \tilde{\beta}_0^m + C_{2m+2} (\hat{\lambda}_1 + C_{2m+1} \hat{\alpha}_0) - \tilde{\beta}_0 - \tilde{\beta}_1 - \tilde{\beta}_2^m - \tilde{\beta}_2^{1+m} - \tilde{\beta}_2^m$.

By Lemma 2.2–2.5 and simple calculation we can obtain the present conclusions. \( \Box \)

3 Main theorem

Consider $2\pi$-periodic equation

\[
\frac{dr}{d\theta} = r(P_1(\cos \theta, \sin \theta)r + P_m(\cos \theta, \sin \theta)r^m) + P_{2m+1}(\cos \theta, \sin \theta)r^{2m+1},
\]

(3.1)

where $P_k(\cos \theta, \sin \theta)$ is homogeneous polynomial in $\cos \theta, \sin \theta$ of degree $k$ and $P_1 \neq 0$. In the case of $m = 1$ [21] I have studied its center-focus problem and proved that for this equation the composition conjecture is correct. In the following we only consider integer $m > 1$.

**Theorem 3.1.** Suppose that $\delta_k^{m+k} + \beta_k^{m+k} \neq 0$ ($0 \leq k \leq m - 1$) and $m + k$ is an odd number, $\delta_k^{2m+1} + \beta_k^{2m+1} + \tilde{\beta}_k^{2m+1} \neq 0$. Then $r = 0$ is a center of (3.1), if and only if

\[
\int_0^{2\pi} P_i P_m d\theta = 0, \quad (i = 0, 1, 2, \ldots, m), \quad \int_0^{2\pi} P_i^{2m+1} P_{2m+1} d\theta = 0, \quad (j = 0, 1, 2, \ldots, m). \quad (3.2)
\]

Moreover, this center is a CC-center. Where the expressions of $\delta_k^i$ and $\beta_k^i$ and $\tilde{\beta}_k^i$ are given in Lemma 2.3, Lemma 2.5 and Lemma 2.7, respectively.

Proof:

**Necessity:**

Let $r(\theta, c)$ be the solution of (3.1) such that $r(0, c) = c$ ($0 < c \ll 1$). We write

\[
r(\theta, c) = c \sum_{n=0}^{\infty} a_n(\theta)c^n,
\]

where $a_0(0) = 1$ and $a_n(0) = 0$ for $n > 1$. The origin of (3.1) is a center if and only if

$r(\theta + 2\pi, c) = r(\theta, c)$, i.e., $a_0(2\pi) = 1$, $a_n(2\pi) = 0$ ($n = 1, 2, 3, \ldots$) [5,6].

Substituting $r(\theta, c)$ into (3.1) we obtain

\[
\sum_{n=0}^{\infty} a'_n(\theta)c^n = cP_1(\theta) \left( \sum_{n=0}^{\infty} a_n(\theta)c^n \right)^2 + c^m P_m \left( \sum_{n=0}^{\infty} a_n(\theta)c^n \right)^{m+1} + c^{2m+1} P_{2m+1}(\theta) \left( \sum_{n=0}^{\infty} a_n(\theta)c^n \right)^{2m+2}. \quad (3.3)
\]

Equating the corresponding coefficients of $c^n$ of (3.3) yields

\[
a'_0(\theta) = 0, \quad a_0(0) = 1,
\]
\( a'_k = P_1 \sum_{i+j=k-1} a_i a_j, \quad a_k(0) = 0, \quad (k = 1, 2, \ldots, m-1), \)

solving these equations we get
\[
a_0 = 1, \quad a_k = P^k_1, \quad (k = 1, 2, \ldots, m-1).
\]

\( a'_{m+k} = P_1 \sum_{i+j=m+k-1} a_i a_j + P_m \sum_{i_1+i_2+\ldots+i_{m+1}=k} a_{i_1} a_{i_2} \ldots a_{i_{m+1}}, a_{m+k}(0) = 0, \quad (k = 0, 1, 2, \ldots, m-1), \)

by these equations we have
\[
a_{m+k} = P^m+k_1 + h_k, \quad (k = 0, 1, 2, \ldots, m-1), \tag{3.4}
\]

where
\[
h'_k = 2P_1 \sum_{i+j=k-1} \bar{P}^i_1 h_j + P_m C^m_{m+k} P^k_1,
\]

by Lemma 2.2, we get
\[
h_k = \sum_{j=0}^{k} h^j_k \bar{P}^{k-j} \sum_{i+j=m} \bar{P}^i_1 P_m, \quad (k = 0, 1, 2, \ldots, m-1),
\]

in which
\[
h^j_k = (k-j+1)C^j_{m+j-2}, \quad (j = 0, 1, 2, \ldots, k-1), \quad h^k_k = C^k_{m+k-2}.
\]

Using (3.3) we have
\[
a'_{2m} = P_1 \left( 2mP^{2m-1}_1 + 2 \sum_{i+j=m-1} \bar{P}^i_1 h_j \right) + P_m (C^m_{2m} \bar{P}^m_1 + C^1_{m+1} h_0),
\]

solving this equation we get
\[
a_{2m} = \bar{P}^{2m}_1 + h_m + a_0, \tag{3.5}
\]

where
\[
h_m = \sum_{j=0}^{m} h^j_m \bar{P}^{m-j} \sum_{i+j=m} \bar{P}^i_1 P_m, \quad a_0 = \frac{m+1}{2} \bar{P}^2_m,
\]

\[
h^j_m = (m-j+1)C^j_{m+j-2}, \quad (j = 0, 1, 2, \ldots, m-1), \quad h^m_m = C^m_{2m-2}.
\]

As \( C^k_{m+k-1} \neq 0 \quad (k = 0, 1, 2, \ldots, m), \) thus, by (3.4) and (3.5) from \( a_{m+k}(2\pi) = 0, \quad (k = 0, 1, 2, \ldots, m) \) follow that
\[
\int_0^{2\pi} \bar{P}^k_m P_m d\theta = 0, \quad (k = 0, 1, 2, \ldots, m). \tag{3.6}
\]

By Lemma 2.1 we get
\[
P_m = P_1 \sum_{i=1}^{m} i\lambda_i \bar{P}^{i-1}_1, \tag{3.7}
\]

therefore,
\[
\bar{P}_m = \lambda_m \bar{P}^m_1 + \sum_{i=1}^{m-1} \lambda_i \bar{P}^i_1 \bar{P}^m_1 P_m = \frac{m \lambda_m}{m+j} + \sum_{i=1}^{m-1} \frac{i}{i+j} \bar{P}^{m+j}_1, \tag{3.8}
\]
where \( \lambda_i \) (\( i = 1, 2, \ldots, m \)) are real numbers. By this we obtain

\[
h_k = \hat{h}_k p_{1}^{m+k} + \sum_{j=1}^{k-1} \hat{h}_{kj} p_{1}^{m+j}, \quad \hat{h}_k = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} h_j, \quad (k = 0, 1, 2, \ldots, m), \tag{3.9}
\]

where \( \hat{h}_{kj} \) (\( j = 1, 2, \ldots, k - 1 \)) are real numbers.

Denote

\[
a_0 = \hat{a}_0 \beta_1^{2m} + \sum_{j=2}^{2m-1} \hat{a}_0 \beta_1^j, \quad \hat{a}_0 = \frac{m+1}{2} \lambda_m^2,
\]

where \( \hat{a}_0 \) (\( j = 2, 3, \ldots, 2m - 1 \)) are real numbers.

Thus,

\[
f = \sum_{i=0}^{\infty} a_i(\theta) c^i = g + c^m h + \alpha c^{2m} + \delta c^{2m+1} + \beta c^{3m+1},
\]

where

\[
g = \sum_{i=0}^{\infty} \beta_1^i c^i, \quad h = \sum_{i=0}^{\infty} h_i c^i, \quad \alpha = \sum_{i=0}^{\infty} \alpha_i c^i, \quad \delta = \sum_{i=0}^{\infty} \delta_i c^i, \quad \beta = \sum_{i=0}^{\infty} \beta_i c^i.
\]

Using (3.3) and (3.11)–(3.15), for \( 0 \leq k \leq m - 1 \), we have

\[
\alpha_{2m+1+k} = P_1 \left( (2m+1+k) \beta_1^{2m+k} + 2 \sum_{i+j=m+k} \beta_1^i h_j + \sum_{i+j=k} (h_i h_j + 2 \beta_1^i \alpha_j) + 2 \sum_{i+j=k-1} \beta_1^i \delta_j \right) 
\]

\[
+ P_m \left( C_{2m+k+1}^{m+k+1} + C_{m+1}^{m+1} \sum_{i+j=k+1} C_{m+i-1}^{m-1} \beta_1^i h_j \right)
\]

\[
+ P_m \sum_{i+j=k-m+1} \left( C_{2m+1}^{2m+1} C_{m+i-2}^{m-2} \beta_1^i \sum_{i_1+i_2=j} h_{i_1} h_{i_2} + C_{m+1}^{m+1} C_{m+i-1}^{m-1} \beta_1^i \alpha_j \right) + P_{2m+1} C_{2m+k+1}^{2m+1} \beta_1^k.
\]
solving this equation we get

\[ a_{2m+1+k} = B^2_{m+k+1} + h_{m+k+1} + \alpha_{k+1} + \delta_k, \quad (k = 0, 1, 2, \ldots, m - 1), \quad (3.16) \]

where \( h_{m+k+1} \) is the solution of equation (2.2), in which \( m + k + 1 \) taking the place of \( k \), \( \delta_k \) is the solution of equation (2.9), \( \alpha_{k+1} \) is the solution of equation (2.11), in which \( k + 1 \) taking the place of \( k \). Thus, by Lemma 2.2, Lemma 2.3 and Lemma 2.4 we get

\[ h_{m+k+1} = \sum_{j=0}^{m+k} h^{j+1}_{m+k+1} h^j_{m+k+1} = h_m + 1 \cdot \ldots, \quad (3.17) \]

where

\[ h^j_{m+k+1} = \frac{2}{m + k + 1 - j} \sum_{i=j}^{m+k} h^j_i = (m + k - j)C^j_{m+j-2}, \quad (i = 0, 1, 2, \ldots, m + k), \]

\[ h^j_{m+k+1} = C^m_{m+k+1} \sum_{j=0}^{m+k} h^j_{m+k+1} = C^{m-2}_{m+k-1} \delta_k + \lambda_m \sum_{j=0}^{m+k+1} \frac{m}{m + j} h^j_{m+k+1}. \]

\[ \alpha_{k+1} = \hat{\alpha}_{k+1} + \delta_{k+1} + \ldots \quad (3.18) \]

\[ \hat{\delta}_k = \sum_{j=0}^{k} \frac{\delta^j_k}{P^{2m+1} \cdot P^{2m+1}} \]

\[ \delta^j_k = \frac{2}{k-j} \sum_{i=j}^{k-1} \delta^j_i = (k - j + 1)C^j_{2m+j-1}, \quad (j = 0, 1, 2, \ldots, k - 1), \]

\[ \delta^k_k = C^k_{2m+k-1} - \sum_{j=0}^{k-1} \delta^j_k = C^k_{2m+k-1}. \]

By (3.17) and (3.18) we see that \( h_{m+k+1} \) and \( \alpha_{k+1} \) are polynomials on \( P_1 \) of degree \( 2m + k + 1 \), so they are \( 2\pi \)-periodic functions. As \( \delta_k = C^k_{2m+k-1} \neq 0 \), so from (3.16) we see that if \( a_{2m+1+k}(2\pi) = 0 \) then

\[ \int_0^{2\pi} P^{2m+1}_1 d\theta = 0, \quad (k = 0, 1, 2, \ldots, m - 1). \quad (3.20) \]
Using (3.3) and (3.11)–(3.15), for $0 \leq k \leq m - 2$, we have

$$a'_{3m+k+1} = P_1((3m + k + 1)P_{3m+k} + 2 \sum_{i+j=2m+k} P_i^j h_j + \sum_{i+j=m+k} (h_i h_j + 2P_i^j \alpha_j)$$

$$+ 2 \sum_{i+j=m+k-1} P_i^j \delta_j + 2 \sum_{i+j=k} h_i \alpha_j + 2 \sum (P_i^j \beta_j + h_i \delta_j))$$

$$+ P_m \left( C^m_{3m+k+1} P_{2m+k+1} + C^1_{m+1} \sum_{i+j=m+k+1} C^{m-1}_{m+i-1} P_i^j h_j + C^2_{m+1} \sum_{i+j=k+1} C^{m-2}_{m+i-1} P_i^j h_j h_iight)$$

$$+ C^1_{m+1} \sum_{i+j=k} C^{m-1}_{m+i-1} P_i^j \alpha_j + C^1_{m+1} \sum_{i+j=k} C^{m-1}_{m+i-1} P_i^j \delta_j$$

$$+ P_{2m+1} (C^2_{3m+k+1} P_{m+k} + C^2_{2m+1} \sum_{i+j=k} C^2_{2m+i} P_i^j h_j),$$

solving this equation we get

$$a_{3m+k+1} = P_1^{3m+k+1} + h_{2m+k+1} + \alpha_{m+k+1} + \delta_{m+k} + \beta_k,$$

(3.21)

where $h_{2m+k+1}$ is the solution of equation (2.2), in which $2m + k + 1$ taking the place of $k$, $\delta_{m+k}$ is the solution of equation (2.9), in which $m + k$ taking place of $k$, $\alpha_{m+k+1}$ is the solution of equation (2.11), in which $m + k + 1$ taking the place of $k$, $\beta_k$ is the solution of the equation (2.14). Thus, by Lemma 2.2, Lemma 2.3 and Lemma 2.4 and Lemma 2.5 we get

$$h_{2m+k+1} = \sum_{j=0}^{2m+k+1} h_j^j = (2m + k + 2 - j)C^j_{m+j-2}, \quad (i = 0, 1, 2, \ldots, 2m + k),$$

(3.22)

where

$$h_j^j = \frac{2}{2m + k + 1 - j} \sum_{i=j}^{2m+k} h_i^j = (2m + k + 2 - j)C^j_{m+j-2}, \quad (i = 0, 1, 2, \ldots, 2m + k),$$

$$h_{2m+k+1} = C^m_{3m+k+1} - \sum_{j=0}^{2m+k} h_j^j = C^2_{3m+k+1},$$

$$h_{2m+k+1} = \lambda_m \sum_{j=0}^{2m+k+1} \frac{m}{m+j} h_j^j,$$

$$\alpha_{m+k+1} = \hat{\alpha}_{m+k+1} P_1^{3m+k+1} + \cdots,$$

(3.23)

$$\hat{\alpha}_{m+k+1} = \frac{\lambda_m^2}{3m + k + 1} \left( \sum_{i+j=m+k} h_i h_j + 2 \sum_{j=0}^{m+k} \hat{\alpha}_j + 2 \sum_{i+j=k} h_i \hat{\alpha}_j + C^1_{m+1} C^2_{m+1} \sum_{i+j=m+1-k} C^m_{m+i-1} \hat{h}_j \hat{h}_iight)$$

$$+ C^1_{m+1} \left( \sum_{i+j=k+1} C^{m-1}_{m+i-1} \hat{\alpha}_j + \sum_{i+j=m+k+1} C^{m-1}_{m+i-1} \hat{h}_j \hat{h}_i \right),$$

$$\delta_{m+k} = \sum_{j=0}^{m+k} \delta_j^j \frac{P_{m+k-j}^j P_{2m+1}^j}{P_1^j P_{2m+1}},$$

(3.24)
\[
\delta^j_{m+k} = \frac{2}{m+k-j} \sum_{i=j}^{m+k-1} \delta^j_i = (m+k-j+1)c^{j}_{2m+j-1}, \quad (j = 0, 1, 2, \ldots, m+k-1),
\]

\[
\delta^{m+k}_{m+k} = \frac{1}{m+k+1} \left( 2 \sum_{j=0}^{m+k-1} \delta^j_0 \right) = \frac{1}{3m+k+1} \left( 3 \sum_{j=0}^{m+k-1} \delta^j_0 \right) = \delta^0_0 = 1.
\]

where

\[
\beta_k = \frac{1}{m+k-j} \left( 2 \sum_{j=0}^{k-1} P_{2m+1} \right) + \beta^j_k \beta^i_{k-j} - \beta^i_{k-j} \beta^j_k - \sum_{j=0}^{k-1} (\beta^i_k + \beta^j_k).
\]

By (3.22) and (3.23) we see that \(h_{2m+1}\) and \(c_{m+1}\) are polynomials on \(\hat{P}_1\) of degree \(3m+k+1\), so they are \(2\pi\)−periodic functions, by (3.21) and (3.24) and (3.25) we see that if

\[
(\delta^{m+k}_{m+k} + \beta^m_{m+k}) \int_0^{2\pi} \hat{P}_1 \hat{P}_2 \, d\theta = 0, \quad (k = 0, 1, 2, \ldots, m-2).
\]

If \(m+k\) is an even integer, then \(\hat{P}_1 \hat{P}_2 \) is an odd polynomial function in \(\cos \theta, \sin \theta\), so

\[
\int_0^{2\pi} \hat{P}_1 \hat{P}_2 \, d\theta = 0. \quad (k = 0, 1, 2, \ldots, m-2).
\]

For \(0 \leq k \leq 2\), using (3.3) and (3.11)–(3.15) we get

\[
a'_{4m+k} = P_1 \left( (4m+k) \hat{P}_{4m+k+1}^4 + 2 \sum_{i+j=3m+k-1} \hat{P}_i^j \hat{h}_j + \sum_{i+j=2m+k-1} (h_i \hat{h}_j + 2 \hat{P}_i^j \alpha_j) + 2 \sum_{i+j=2m+k-2} \hat{P}_i^j \delta_j \right)
\]

\[
+ 2 \sum_{i+j=m+k} (\hat{P}_i^j \alpha_j + 2 \sum_{i+j=2m+k-2} (\alpha_i \alpha_j + 2 \sum_{i+j=k-2} (h_i \beta_j + \alpha_i \delta_j))
\]

\[
+ P_m \left( c_{4m+k} \hat{P}_{4m+k}^3 + c_{m+1} \sum_{i+j=2m+k} c_{m+i-1}^m \hat{P}_i^j h_j + c_m^2 \sum_{i+j=k} c_{m+i-2}^m \hat{P}_i^j h_i h_l \right)
\]
Composition conjecture for rigid systems

Taking the place of where \( \hat{h} \), solving this equation we get

\[ a_{4m+k} = \bar{P}_1^{4m+k} + h_{3m+k} + \alpha_{2m+k} + \bar{\alpha}_k + \delta_{2m+k-1} + \beta_{m+k-1} + \bar{\beta}_k, \quad (k = 0, 1, 2), \tag{3.27} \]

where

\[ h_{3m+k}' = 2P_i \sum_{i+j=3m+k-1} \bar{P}_1^i h_j + P_mC_{4m+k}^3 \bar{P}_1^{3m+k}, \]

by Lemma 2.2 we obtain

\[ h_{3m+k} = \sum_{j=0}^{3m+k} h^j_{3m+k} P_{4m+k-j}^1 \bar{P}_1^j P_m = \hat{h}_{3m+k} P_{4m+k}^1 + \sum_{j=1}^{4m+k-1} \hat{h}_{3m+k-j} P_{4m+k-j}^1, \tag{3.28} \]

in which \( \hat{h}_{3m+k-j} (j = 1, 2, \ldots, 4m + k - 1) \) are real numbers,

\[ h^j_{3m+k} = \frac{2}{3m+k-j} \sum_{i=j}^{3m+k-1} h_i = (3m + k - j + 1)C_{m+j-2}, \quad (j = 0, 1, 2, \ldots, 3m + k - 1), \]

\[ h_{3m+k} = C_{4m+k-1}^m - \sum_{j=0}^{3m+k-1} h^j_{3m+k} = C_{4m+k-2}, \quad \hat{h}_{3m+k} = \lambda_m \sum_{j=0}^{3m+k} \frac{m}{m+j} h^j_{3m+k}. \]

\[ \alpha_{2m+k}' = P_1 \sum_{i+j=2m+k-1} (h_i h_j + 2P_i^j \alpha_j) + 2P_1 \sum_{i+j=2m+k-1} h_i \alpha_j + P_mC_{m+1}^1 \sum_{i+j=2m+k} C_{m+i-1}^{m-2} \bar{P}_1^j h_j P_m \]

\[ + P_mC_{2m+1}^1 \sum_{i+j+l=2m+k} C_{m+i-2}^{m-2} \bar{P}_1^j h_j + P_mC_{m+1}^1 \sum_{i+j+l=2m+k} C_{m+i-1}^{m-2} \bar{P}_1^j \bar{P}_1^j \alpha_j, \]

by Lemma 2.4 we have

\[ \alpha_{2m+k} = \hat{\alpha}_{2m+k} P_{4m+k}^1 + \sum_{j=0}^{4m+k-1} \hat{\alpha}_{2m+k-j} P_{4m+k-j}^1, \]

where \( \hat{\alpha}_{2m+k-j} (j = 0, 1, 2, \ldots, 2m + k - 1) \) are real numbers, \( \hat{\alpha}_{2m+k} \) is expressed by (2.13) with taking the place of \( k \) by \( 2m + k \).
substituting (2.3) and (2.12) into it and solving this equation we get \( \hat{\alpha}_k = \hat{\alpha}_k \hat{p}^{4m+k} + \cdots \),

\[
\hat{\alpha}_k = \frac{1}{4m+k} \sum_{i+j=k-1} \hat{\alpha}_i \hat{\alpha}_j + m \lambda_m \sum_{i+j+l+k} C_{m+i}^{m-3} \hat{\delta}_{i,j} \hat{\delta}_{i} + m \lambda_m C_{m}^{1} C_{m}^{1} \sum_{i+j+l+k} C_{m+i-2}^{m-2} \hat{\delta}_{i,j} \hat{\delta}_{i}.
\]

Thus,

\[
\alpha_{2m+k} + \hat{\alpha}_k = (\hat{\alpha}_{2m+k} + \hat{\delta}_k) \hat{p}^{4m+k} + \cdots, \quad (k = 0, 1, 2).
\]

(3.29)

by Lemma 2.3 we obtain

\[
\delta_{2m+k-1} = \sum_{j=0}^{2m+k-1} \delta_{2m+k-1-j} \hat{p}_{1}^{j} \hat{p}_{2m+k-1}^{j},
\]

(3.30)

by Lemma 2.5, we get

\[
\beta_{m+k-1} = \sum_{j=0}^{m+k-1} \beta_{m+k-1-j} \hat{p}_{1}^{j} \hat{p}_{2m+k-1}^{j},
\]

(3.31)

where \( \beta_{m+k-1} (j = 0, 1, 2, \ldots, m+k-1) \) are expressed by (2.16)–(2.18).

\[
\hat{\beta}_k = 2 \beta_1 \sum_{i+j=k-2} (h_i \beta_j + \alpha_i \delta_j) + C_{m+1}^{1} \beta_m \sum_{i+j=k-1} C_{m+i-1}^{m-1} \hat{p}_{1}^{j} \beta_j
\]

\[
+ P_m C_{m+1}^{1} C_{m+i-2}^{m-2} \hat{p}_{1}^{j} \hat{\delta}_i \hat{\delta}_j + P_{2m+1} \left( \sum_{i+j+l+k} C_{m+i-2}^{m-2} \hat{p}_{1}^{j} \hat{\delta}_i \hat{\delta}_j + P_{2m+1} \left( \sum_{i+j+l+k} C_{m+i-2}^{m-2} \hat{p}_{1}^{j} \hat{\delta}_i \hat{\delta}_j \right) \right), \quad (k = 0, 1, 2),
\]

from this and Lemma 2.7 we get

\[
\hat{\beta}_0 = 0,
\]

(3.32)

\[
\hat{\beta}_1 = \hat{\beta}_1 \hat{p}_{2m+1} \hat{p}_{2m+1} + \hat{\beta}_1 \hat{p}_{1}^{m} \hat{p}_{2m+k-1}^{m} + \hat{\beta}_1 \hat{p}_{1}^{2m} \hat{p}_{2m+1} \hat{p}_{2m+1} + \cdots,
\]

(3.33)

\[
\hat{\beta}_2 = \hat{\beta}_2 \hat{p}_{2m+1} \hat{p}_{2m+1} + \hat{\beta}_2 \hat{p}_{2m+1} \hat{p}_{2m+1} + \hat{\beta}_2 \hat{p}_{2m+1} \hat{p}_{2m+1} + \cdots,
\]
By (3.28) and (3.29) we see that $h_{3m+k}$ and $\alpha_{2m+k} + \bar{\alpha}_k$ are polynomials on $\bar{P}_1$ of degree $4m + k$, so they are $2\pi$-periodic functions. By (2.31) and (3.28) and (3.31) and (3.32) and (3.33) and (3.20) and (3.26) we see that from $a_{4m}(2\pi) = 0$ implies

$$\left(\delta_{2m-1}^{2m-1} + \beta_{m-1}^{2m-1}\right) \int_0^{2\pi} \bar{P}_1^{2m-1} P_{2m+1} d\theta = 0, \tag{3.34}$$

by the hypothesis, $\delta_{2m-1}^{2m-1} + \beta_{m-1}^{2m-1} \neq 0$, thus, from (3.34) we get

$$\int_0^{2\pi} \bar{P}_1^{2m-1} P_{2m+1} d\theta = 0. \tag{3.35}$$

From $a_{4m+2}(2\pi) = 0$ follows

$$\left(\delta_{2m+1}^{2m+1} + \beta_{m+1}^{2m+1} + \bar{\beta}_2^{m+1}\right) \int_0^{2\pi} \bar{P}_1^{2m+1} P_{2m+1} d\theta = 0, \tag{3.36}$$

as $\delta_{2m+1}^{2m+1} + \beta_{m+1}^{2m+1} + \bar{\beta}_2^{m+1} \neq 0$, from (3.36) we get

$$\int_0^{2\pi} \bar{P}_1^{2m+1} P_{2m+1} = 0. \tag{3.37}$$

In summary, by (3.6) and (3.20) and (3.26) and (3.35) and (3.37), the condition (3.2) is necessary for the origin to be a center of equation (3.1).

**Sufficiency.**

Now, we show that the condition (3.2) is also sufficient center condition.

By Lemma 2.1 and (3.2) we have

$$P_m = P_1 \sum_{j=1}^{m} j \lambda_j \bar{P}_1^{j-1}, \quad P_{2m+1} = P_1 \sum_{j=0}^{2m} \mu_j \bar{P}_1^j,$$

where $\lambda_j$, $\mu_j$ are real numbers. Thus by Theorem 1.2, the $r = 0$ is a center of (3.1), moreover this center is CC-center, i.e, under condition of present theorem the composition conjecture is valid for equation (3.1).

In summary, Theorem 3.1 has been proved. \(\square\)

**Remark 3.2.** By Theorem 3.1, we can derive all the focal quantity formulas of system (1.5) with arbitrary $m$ and they contain exactly $\left\lfloor \frac{m}{2} \right\rfloor + m + 2$ relations (3.2) and they are more concise and beautiful than the results calculated by computer.

**Corollary 3.3.** If $m > 1$ and $\lambda_m \geq 0$, then the $r = 0$ is a center of (3.1), if and only if the Moments Conditions:

$$\int_0^{2\pi} \bar{P}_1^i P_m d\theta = 0, \quad (i = 0, 1, 2, \ldots, m), \quad \int_0^{2\pi} \bar{P}_1^{2j+1} P_{2m+1} d\theta = 0, \quad (j = 0, 1, 2, \ldots, m).$$

are satisfied.

**Proof.** By Lemma 2.3 we see $\delta^i > 0$, from Lemma 2.6 follows that $\beta_m^{n+k} \geq 0$, using Lemma 2.7 we have $\bar{P}_2^{2m+1} \geq 0$, so $\bar{P}_m^{m+k} + \beta_k^{m+k} \neq 0$ $(0 \leq k \leq m - 1)$ and $\delta_{2m+1}^{2m+1} + \beta_{m+1}^{2m+1} + \beta_2^{2m+1} \neq 0$, by Theorem 3.1 the conclusion of the present corollary is true. \(\square\)
Consider equation
\[ r' = r(P_1r + P_2r^2 + P_3r^5), \] (3.38)
where \( P_k = \sum_{i+j=k} p_{ij} \cos^i \theta \sin^j \theta \), \((k = 1, 2, 5)\) and \( P_1 \neq 0 \) and \( p_{ij} \) are real numbers.

In (2.6) taking \( m = 2 \), we get
\[ h_0 = \lambda_2, \quad h_1 = \frac{8}{3} \lambda_2, \quad h_2 = \frac{29}{6} \lambda_2, \]
by [21], \( \lambda_2 = \frac{-p_{10}}{2p_{10}p_{01}} \) (if \( p_{10}p_{01} \neq 0 \)) or \( \lambda_2 = \frac{p_{11}}{2(p_{10} - p_{01})} \) (if \( p_{10}^2 - p_{01}^2 \neq 0 \)). In Lemma 2.3, taking \( m = 2, k = 3, k = 5 \) we obtain
\[ \delta_3^2 = 20, \quad \delta_5^2 = 56. \]

In (2.26) taking \( m = 2, k = 1, 2, 3 \), we get
\[ \beta_0^2 = 3 \lambda_2, \quad \beta_1^2 = \frac{52}{3} \lambda_2, \quad \beta_2^2 = \frac{117}{2} \lambda_2, \quad \beta_3^2 = \frac{452}{3} \lambda_2, \quad \beta_5^2 = 56 \lambda_2^2. \]

Theorem 3.1 implies the following corollary.

**Corollary 3.4** (23). If \( (5 + \frac{13}{3} \lambda_2)(14 + \frac{113}{9} \lambda_2 + 14 \lambda_2^2) \neq 0 \), then the \( r = 0 \) is a center of (3.38) if and only if,
\[ \int_0^{2\pi} P_1^{2i} P_2 d\theta = 0, \quad (i = 0, 1), \quad \int_0^{2\pi} P_1^{2i+1} P_3 d\theta = 0, \quad (j = 0, 1, 2), \] (3.39)
where \( \lambda_2 = \frac{-p_{10}}{2p_{10}p_{01}} \) (if \( p_{10}p_{01} \neq 0 \)) or \( \lambda_2 = \frac{p_{11}}{2(p_{10} - p_{01})} \) (if \( p_{10}^2 - p_{01}^2 \neq 0 \)).

**Remark 3.5.** By Corollary 3.4 follows that if \( (5 + \frac{13}{3} \lambda_2)(14 + \frac{113}{9} \lambda_2 + 14 \lambda_2^2) \neq 0 \), then system
\[ \begin{cases} x' = y + x(\sum_{i+j=1} p_{ij}x^i y^j + \sum_{i+j=2} p_{ij}x^i y^j + \sum_{i+j=5} p_{ij}x^i y^j), \\ y' = x + y(\sum_{i+j=1} p_{ij}x^i y^j + \sum_{i+j=2} p_{ij}x^i y^j + \sum_{i+j=5} p_{ij}x^i y^j) \end{cases} \]
has a center at \((0, 0)\), if and only if its five focal values are equal to zero, i.e.,
\[ p_{20} + p_{02} = 0, \]
\[ p_{20}(p_{01}^2 - p_{10}^2) - p_{11}p_{10}p_{01} = 0, \]
\[ p_{20}(5p_{50} + p_{32} + p_{14}) - p_{10}(5p_{50} + p_{23} + p_{41}) = 0, \]
\[ p_{10}^3(p_{23} + 10p_{05}) - 3p_{10}p_{01}(2p_{14} + p_{32}) + 3p_{10}p_{01}(p_{23} + 2p_{41}) - p_{01}(p_{32} + 10p_{50}) = 0, \]
\[ p_{50}p_{01}^3 - p_{41}p_{01}^3p_{10} + p_{32}p_{01}^3p_{10} - p_{23}p_{01}^3p_{10} + p_{41}p_{10}^3p_{01} - p_{05}p_{10}^3 = 0. \]

Taking \( p_{10} = p_{01} = 1, p_{20} = -2, p_{11} = 0, p_{02} = 2, p_{50} = p_{05} = a, p_{41} = p_{14} = b, p_{23} = p_{32} = c, \lambda_2 = 1 \), the above conditions are satisfied, from Remark 3.5 we get the following example.

**Example 3.6.** Differential system
\[ \begin{cases} x' = -y + x(x + y - 2x^2 + 2y^2 + ax^5 + bx^4y + cx^3y^2 + cx^2y^3 + bxy^4 + ay^5), \\ y' = x + y(x + y - 2x^2 + 2y^2 + ax^5 + bx^4y + cx^3y^2 + cx^2y^3 + bxy^4 + ay^5) \end{cases} \]
has a CC-center at \((0, 0)\). Here \( a, b, c \) are arbitrary numbers.
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References


